THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: Randal Lee Kobes

TITLE OF THESIS: Interacting Spin 3/2 Theories

DEGREE FOR WHICH THESIS WAS PRESENTED: M. Sc.

YEAR THIS DEGREE GRANTED: 1980

THE UNIVERSITY OF ALBERTA

INTERACTING SPIN 3/2 THEORIES

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

FALL, 1980

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled INTERACTING SPIN 3/2 THEORIES submitted by Randal Lee Kobes in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The inconsistencies present when interactions are introduced into a spin 3/2 Lagrangian are examined. At the c-number level, there exists acausal propagation, while in the quantized version, certain equal-time anticommutators are indefinite.

First considering the case of minimal coupling, we attempt to correct the situation by addition of a scalar and spin 1/2 field and then by a spin 1/2 field alone to the Lagrangian. In both cases the problems remain.

We then consider the Bhabha-Gupta theory of a mixed spin 3/2 - spin 1/2 field with two different mass states. Causality here may be preserved if the total free charge density is allowed to be indefinite. An indefinite metric is thus needed upon quantization, where we find the Johnson-Sudarshan problem with the anticommutators disappears for the independent fields. These results can be partially explained by examining the particular choice of the coupling constant, which actually turns the secondary constraint into a primary constraint.

Finally, we attempt to find a physical explanation for the unsatisfactory behaviour of interacting higher spin theories. If one accepts the view that all particles with spin $s \geq 3/2$ are not elementary but composite, then such an explanation may be possible in light of the fact that the Hamiltonian in the corresponding Schrödinger equation

is non-local for higher spin fields. However, at least in the mind of the author, the question as to whether or not we have put enough information into the theory initially to justify this is still open.

NOTATION

In order to avoid possible confusion, the conventions used in this work will now be described.

Natural units, in which $\hbar = c = 1$, are used throughout.

A four vector has the components

$$x = (x^o, \vec{x}) \ .$$

Summation over repeated indices is understood. Greek indices assume values from 0 to 3, while Latin ones run from 1 to 3. Capital Latin spinor indices, used only in chapter II, assume values from 1 to 2.

The metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1) \ .$$

The scalar product of 2 four vectors is thus

$$x \cdot y = g_{\mu\nu} x^\mu x^\nu = x^o y^o - \vec{x} \cdot \vec{y} \ .$$

We assume the conventions

$$p_\mu = i\partial_\mu$$

$$\pi_\mu = p_\mu + eA_\mu \ .$$

As well, for two operators A and B, we define

$$[A,B] = AB - BA$$

$$\{A,B\} = AB + BA \ .$$

The four Dirac matrices γ^μ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad .$$

When required, the following representation is employed:

$$\gamma^o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

$$\gamma_5 = \gamma^o\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where σ^k are the 2×2 Pauli matrices.

ACKNOWLEDGEMENTS

I would like to thank Dr. A.Z. Capri for introducing me to this problem and to field theory in general. His guidance has been instrumental in producing this thesis.

Discussions with fellow graduate students John Whitehead and Gordon Semenoff have helped to overcome some troublesome points in the course of this work.

I am indebted to the Natural Sciences and Engineering Research Council of Canada for financial support over the last two years. I also wish to thank Mrs. M. Yiu for her careful and accurate typing of this thesis.

Finally, to my parents, for providing a retreat away from this madness.

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CHAPTER I

STATEMENT OF THE PROBLEM

This work is concerned with problems associated with the theory of spin 3/2 fields. Although the free field case is devoid of problems, inconsistencies arise when interactions are introduced.

In chapter II we present the theory of spin 3/2 fields without interaction. The pioneering work here was done by Dirac, Fierz and Pauli whose formulation was in terms of spinors. We then give the more easily handled vector-spinor theory as proposed by Rarita and Schwinger.

Chapter III explores the inconsistencies present when interactions are introduced. We first concentrate on the minimally coupled Rarita-Schwinger (RS) field with an external electromagnetic field. Interpreted as a c-number theory, there exists acausal propagation, while indefinite equal-time anticommutation relations are found in the q-number theory. The same two problems also occur for the RS field coupled to a spinor and a scalar field.

In chapter IV we attempt two modifications of the RS field with minimal coupling. We first try a combination of the two interactions discussed in chapter III, but find the undesirable effects do not cancel. The two problems also remain when a spin 1/2 field is added to the RS field by scalar coupling.

In chapter V we try coupling the RS field to a spin 1/2 field by derivative coupling. This system is in fact the Bhabha-Gupta (BG) theory of a mixed spin 3/2 - spin 1/2 field with two different mass states. Causality in both the case of electromagnetic and scalar-spinor interactions can be retained here, but at the expense of having an indefinite total charge for the free field. Upon quantization, we find this indefiniteness manifests itself as an indefinite metric being forced upon us if we wish to eliminate the anticommutator problem for the independent fields. We then discuss the significance of the particular choice made for the coupling constant.

We finally give some concluding remarks in chapter VI, with a general outlook on the area of higher spin theories being offered.

CHAPTER II

FORMULATION OF SPIN 3/2 THEORIES

A. The Equations of Dirac

The discovery of the Dirac equation for particles of spin 1/2 and its application to the case of electrons prompted the search for equations of higher spin fields. Although not compelled by experimental evidence for a theoretical description, there was no reason to deny the existence of higher spin particles that might be found in the future. As well, such a set of equations would present a more unified view of the concept of spin in relativistic quantum field theory, thereby putting the theory on a more logically complete foundation.

Dirac [1936] was the first to give a systematic procedure for constructing field equations of higher spin particles. The basic field operators used are spinors of various degree, according to the spin considered (for some basic properties of spinors, see Appendix I). For spin 3/2, two symmetric spinors

$$\psi_{AB}^{\dot{C}} = \psi_{BA}^{\dot{C}}$$

$$\phi_{C}^{\dot{A}\dot{B}} = \phi_{C}^{\dot{B}\dot{A}}$$

are used. They satisfy the equations

$$p^{\dot{B}C}\,\psi_{CD}^{\dot{A}} + p^{\dot{A}C}\,\psi_{CD}^{\dot{B}} = 2m\,\phi_{D}^{\dot{A}\dot{B}} \tag{2.1a}$$

$$p_{A\dot{C}}\,\phi_{B}^{\dot{C}\dot{D}} + p_{B\dot{C}}\,\phi_{A}^{\dot{C}\dot{D}} = 2m\,\psi_{AB}^{\dot{D}} \tag{2.1b}$$

subject to the conditions

$$p_{\dot{A}}^{\ B}\,\psi_{BC}^{\dot{A}} = 0 = p_{\dot{A}}^{\ B}\,\phi_{B}^{\dot{A}\dot{C}} \quad . \tag{2.2}$$

The constraints (2.2) impose eight conditions on the 16 components of the field, leaving eight components to describe a spin 3/2 field. The two fields eliminated are described by spin 1/2 for rotations in spin space. This elimination is necessary to ensure that the total charge density is positive definite, thereby enabling the field to be quantized in the usual manner. However, the spin 1/2 fields are needed if the total energy is to be positive definite. We shall encounter this difficulty again when we consider the Bhabha equation for a mixed spin 3/2 - spin 1/2 system [Bhabha, 1952].

B. The Fierz-Pauli Modification

Electromagnetic coupling in Dirac's equations was assumed to be accomplished by the minimal substitution

$$\partial_\mu \rightarrow \partial_\mu - i\,e\,A_\mu$$

where A_μ are the electromagnetic potentials. However, Fierz and Pauli [1939] observed that an algebraic inconsistency arose if this prescription was used. In effect, a relation derived from one of (2.1) was incompatible with (2.2).

To solve this dilemma, they suggested that the field equations (2.1) and (2.2) be derived from a Lagrangian by a variational principle. Two auxiliary fields were introduced such that the desired equations were obtained in the free field case. With no interaction terms present, these auxiliary fields vanished identically. Upon introduction of minimal coupling, the constraints that followed in general involved the auxiliary fields. The eight independent components left upon use of these constraints then described a pure spin 3/2 state. The difficulties found in Dirac's equations were eliminated, and a consistent method for describing charged spin 3/2 fields, as well as those of higher spin, was thought to be available.

C. The Rarita-Schwinger Formalism

Another formulation of the theory of spin 3/2 fields was given by Rarita and Schwinger [1941]. In this approach, the fields are described by a 16 component vector spinor ψ_μ, with each of the four vector indices representing a Dirac four-spinor. This feature, together with the fact that the Lagrangian is composed of the Dirac matrices, makes manipulations easier in this formalism.

i) The Nature of Constraints

The Lagrangian is given by

$$\mathcal{L} = \bar{\psi}^\mu (\Gamma^\sigma)_\mu{}^\nu p_\sigma \psi_\nu - \mathcal{H} \tag{2.3}$$

where

$$\bar{\psi}^{\mu} = \psi^{\mu\dagger}\gamma^{o}$$

$$p_{\sigma} = i\partial_{\sigma}$$

and $\mathcal{H}$ is an invariant function of the field ψ_{μ} and any additional fields to which it is coupled. The four 16×16 matrices $(\Gamma^{\sigma})_{\mu}^{\ \nu}$ are Hermitian in the sense that

$$[\gamma^{o}(\Gamma^{\sigma})_{\mu}^{\ \nu}]^{\dagger} = \gamma^{o}(\Gamma^{\sigma})^{\nu}_{\ \mu} \equiv (A^{\sigma})^{\nu}_{\ \mu} \quad .$$

The Euler-Lagrange equations following from (2.3) are

$$(A^{\sigma})_{\mu}^{\ \nu} p_{\sigma}\,\psi_{\nu} = \gamma^{o}\ \frac{\delta\mathcal{H}}{\delta\bar{\psi}^{\mu}} \tag{2.4}$$

where

$$\frac{\delta\mathcal{H}}{\delta\bar{\psi}^{\mu}} \equiv \frac{\partial\mathcal{H}}{\partial\bar{\psi}^{\mu}} - p_{\nu}\ \frac{\partial\mathcal{H}}{\partial(p_{\nu}\bar{\psi}^{\mu})} \quad .$$

It may occur that the matrix A^{o} is singular, in which case constraints arise in the following manner. (Constraints here will be taken to mean relations among the field components which do not involve time derivatives.)

Let P_{o} be the projection matrix associated with the zero eigenvalue of A^{o}, so that $P_{o}A^{o} = 0$, with $(P_{o})^{2} = P_{o}$. Operating on (2.4) with P_{o} then gives the constraints

$$P_{o}(A^{k})_{\mu}^{\ \nu} p_{k}\,\psi_{\nu} = P_{o}\,\gamma^{o}\ \frac{\delta\mathcal{H}}{\delta\bar{\psi}^{\mu}} \quad . \tag{2.5}$$

Now, by considering the invariance of equation (2.4) under an infinitesimal Lorentz transformation,

$$x_\mu \to x'_\mu = x_\mu + \varepsilon_\mu{}^\nu x_\nu$$

$$\psi(x) \to \psi'(x') = (1 + \varepsilon^{\mu\nu} S_{\mu\nu})\psi(x) \ ,$$

we obtain the relation

$$[\Gamma^\sigma, S_{\mu\nu}] = i(\Gamma_\mu g_\nu{}^\sigma - \Gamma_\nu g_\mu{}^\sigma) \ .$$

In particular,

$$\Gamma^o S_{ok} - S_{ok}\Gamma^o = -i\Gamma_k$$

$$A^o S_{ok} - \gamma^o S_{ok} \gamma^o A^o = -iA_k \ .$$

We thus arrive at

$$P_o A_k P_o = 0$$

so we can write (2.5) as

$$P_o (A^k)_\mu{}^\nu p_k (1 - P_o) \psi_\nu = P_o \gamma^o \frac{\delta \mathcal{H}}{\delta \bar{\psi}^\mu} \ . \qquad (2.6)$$

These are called primary constraints as they result from the kinematics of the field (i.e. the singular nature of A^o), the components $P_o\psi$ being independent of the dynamical term $\mathcal{H}$. They are to be distinguished from possible secondary constraints which define another set of components whose structure will depend on $\mathcal{H}$.

Since the left side of (2.6) does not involve the components $P_o\psi$, two possibilities arise. One is that the structure of the right side is such that all of the

components $P_o\psi$ are expressed in terms of $(1-P_o)\psi$ by (2.6). Substituting (2.6) into (2.4) will then give the equation of motion for the independent components $(1-P_o)\psi$. The other possibility is that (2.6) leaves some or all of the components $P_o\psi$ undetermined, giving instead additional constraints on $(1-P_o)\psi$. When (2.6) is then used in conjunction with (2.4), new secondary constraints will emerge which involve the dynamical term $\mathcal{H}$.

In their classic paper, Johnson and Sudarshan [1961] showed that secondary constraints must occur if the field is to be quantized according to Fermi-Dirac statistics and the spin of the field is greater than 1/2. To do this, they first constructed the generator of infinitesimal transformations of the field ψ (see Schwinger [1953] and Roman [1969]),

$$\delta\psi = i[G,\psi]$$

where

$$G = i\int \left(\frac{\partial\mathcal{L}}{\partial(p_o\psi_\mu)}\right)\delta\psi_\mu\, d^3y = i\int\psi^{\mu\dagger}(A^o)_\mu{}^\nu\,\delta\psi_\nu d^3y\ . \qquad (2.7)$$

If there were no constraints on the components $\tilde{\psi} \equiv (1-P_o)\psi$, then with the aid of this generator the following equal-time anticommutation relations (ETACR) could be derived:

$$\{\tilde{\psi}^\dagger_\alpha(x),\tilde{\psi}_\beta(y)\}\delta(x_o-y_o) = [(1-P_o)A_o(1-P_o)]^{-1}_{\alpha\beta}\,\delta(\vec{x}-\vec{y})\ . \qquad (2.8)$$

The matrix A^o is to be inverted in its nonsingular subspace. Since the left side of (2.8) is positive

definite, the right side must also be. However, as Johnson and Sudarshan [1961] showed, A^o is positive definite only if it is a multiple of the unit matrix and the spin of the field is 1/2; otherwise, it is indefinite. Thus, for theories of spin 3/2 or greater, there must exist additional (secondary) constraints on the components $(1 - P_o)\psi$ so that (2.8) does not follow from (2.7). This is because only independent components, not related by constraints, are allowed in the expression for the generator.

ii) The Lagrangian

Having discussed the nature of constraints, we now proceed with the formulation of the Rarita-Schwinger (RS) Lagrangian. The most general form of the matrices $(\Gamma^\sigma)_\mu{}^\nu$, consistent with Hermiticity and a description of Fermi-Dirac statistics, is

$$-(\Gamma^\sigma)_\mu{}^\nu = \gamma^\sigma g_\mu{}^\nu + W(g_\mu{}^\sigma\gamma^\nu + g^{\nu\sigma}\gamma_\mu) + K\gamma_\mu\gamma^\sigma\gamma^\nu$$

where W and K are real parameters. The matrix $A^o = \gamma^o\Gamma^o$ has eigenvalues x satisfying

$$(x-1)^8[x^2 + 2(W+2K)x + (2K - 3W^2 - 2W - 1)]^4 = 0 \ ,$$

so is indefinite as required. It may be made singular by taking

$$K = \frac{1}{2}(3W^2 + 2W + 1)$$

in which case the eigenvalues solve

$$x^4(x-1)^8[x + 2(3W^2 + 3W + 1)]^4 = 0 \ .$$

The eigenvectors of the eigenvalues 0 and $-2(3W^2 + 3W + 1)$ will be seen to transform as spin 1/2 spinors, but that corresponding to the eigenvalue 1 will not simply be described as a spin 3/2 object.

The parameter W is still left undetermined. To find its significance, consider the point transformation of the field components

$$\psi_\mu \rightarrow \psi_\mu' = \psi_\mu + a\gamma_\mu \gamma\cdot\psi \quad .$$

The transformed Lagrangian then has exactly the same form as (2.3), with W being transformed as

$$W \rightarrow W' = W(1 + 4a) + 2a \quad .$$

Except for $a = -1/4$, when the transformation is singular, this has no bearing on the structure of the set of components described by spin 3/2. The value of W simply determines the two sets of spin 1/2 components, and is without physical significance.

In the following, we choose $W = -1/3$, fixing the eigenvalues corresponding to the two spin 1/2 components to be 0 and -2/3. The projection matrices are then

$$(P_o)_\mu{}^\nu = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \gamma_1\gamma^1 & \gamma_1\gamma^2 & \gamma_1\gamma^3 \\ 0 & \gamma_2\gamma^1 & \gamma_2\gamma^2 & \gamma_2\gamma^3 \\ 0 & \gamma_3\gamma^1 & \gamma_3\gamma^2 & \gamma_3\gamma^3 \end{pmatrix}$$

$$(P_{-2/3})_\mu{}^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The two spin 1/2 components are therefore $\vec{\gamma}\cdot\vec{\psi}$ and ψ_o, respectively. The spin 3/2 components are described by the set

$$\begin{aligned}\phi_k &\equiv (g - P_o - P_{-2/3})_\mu{}^\nu \psi_\nu \\ &= (g_k{}^\ell - \frac{1}{3}\gamma_k\gamma^\ell)\psi_\ell \quad .\end{aligned}$$

We can write for the term $\mathcal{H}$ in (2.3)

$$\mathcal{H} = \mathcal{H}' - \bar{\psi}^\mu B_\mu{}^\nu \psi_\nu$$

where

$$B_\mu{}^\nu = -m(g_\mu{}^\nu + T\gamma_\mu\gamma^\nu) \ .$$

The primary constraint (2.6) becomes

$$P_o(A^k)_\mu{}^\nu p_k(1-P_o)\psi_\nu + P_o\gamma^o B_\mu{}^\nu\psi_\nu = P_o\gamma^o \frac{\delta\mathcal{H}'}{\delta\bar{\psi}^\mu}$$

or

$$P_o[(A^k)_\mu{}^\nu p_k + \gamma^o B_\mu{}^\nu](1-P_o)\psi_\nu + P_o\gamma^o B_\mu{}^\nu P_o\psi_\nu = P_o\gamma^o \frac{\delta\mathcal{H}'}{\delta\bar{\psi}^\mu} \quad .$$

If secondary constraints imposed on the components $(1-P_o)\psi$ are to emerge from this, then $P_o\,\gamma^o B_\mu{}^\nu\,P_o$ must vanish. We find this occurs, if, for $W = -1/3$, T is taken to be -1/3.

Thus, the Lagrangian for the free spin 3/2 field, where $\mathcal{H}' = 0$, can be written in the form

$$\mathcal{L} = \bar{\psi}^{\mu} (\Gamma \cdot p - B)_{\mu}{}^{\nu} \psi_{\nu} \tag{2.9}$$

where

$$(\Gamma^{\sigma})_{\mu}{}^{\nu} = -\gamma^{\sigma} g_{\mu}{}^{\nu} + \frac{1}{3} (\gamma_{\mu} g^{\nu\sigma} + \gamma^{\nu} g_{\mu}{}^{\sigma}) - \frac{1}{3} \gamma_{\mu} \gamma^{\sigma} \gamma^{\nu}$$

$$B_{\mu}{}^{\nu} = -m (g_{\mu}{}^{\nu} - \frac{1}{3} \gamma_{\mu} \gamma^{\nu}) \ .$$

iii) Introduction of Minimal Coupling

The comments made concerning minimal coupling in the Dirac spinor theory also apply in the vector-spinor formalism. With the Lagrangian (2.9), we can derive the equations of motion

$$(\gamma \cdot p - m) \psi_{\mu} - \frac{1}{3} (\gamma_{\mu} p^{\nu} + p_{\mu} \gamma^{\nu}) \psi_{\nu} + \frac{1}{3} \gamma_{\mu} (\gamma \cdot p + m) \gamma \cdot \psi = 0 \ . \tag{2.10}$$

Contracting with γ^{μ} and p^{μ} yields, respectively,

$$m\gamma \cdot \psi = -2p \cdot \psi \tag{2.11}$$

$$mp \cdot \psi = \frac{1}{3} m\gamma \cdot p\gamma \cdot \psi + \frac{2}{3} \gamma \cdot pp \cdot \psi \ . \tag{2.12}$$

Substituting (2.11) into (2.12) leads to the conclusions that, for $m \neq 0$,

$$p \cdot \psi = 0 = \gamma \cdot \psi \ .$$

Thus, the equations (2.10) are equivalent to the system

$$(\gamma \cdot p - m) \psi_{\mu} = 0$$

$$\text{with} \quad p \cdot \psi = 0 = \gamma \cdot \psi \ . \tag{2.13}$$

If we were now to subject (2.13) to minimal coupling by the substitution $p_\mu \to p_\mu + eA_\mu$, we could derive the additional constraint

$$\gamma \cdot F \cdot \psi = 0$$

which is incompatible with the other equations. Hence, minimal coupling must be introduced from the outset in equation (2.10) in order that such algebraic inconsistencies be avoided.

iv) An Equivalent Form of the Lagrangian

Before proceeding with an analysis of the field with electromagnetic coupling, we shall find it advantageous to cast the Lagrangian (2.9) into a form which involves explicitly the spin 3/2 and two spin 1/2 components. This formalism is due to Dr. Y. Takahashi [personal notes].

The equivalent form of the RS Lagrangian is

$$-\mathcal{L} = m_\mu{}^\nu \phi^{\mu\dagger} p_o \phi_\nu + \phi^{\mu\dagger} A_\mu{}^\nu \phi_\nu + \lambda^\dagger C^\mu \phi_\mu + \phi^{\mu\dagger} (C_\mu)^\dagger \lambda \qquad (2.14)$$

where

$$\phi_\mu = [\phi_o, \phi_k]$$

$$= [\psi_o, (g_k{}^\ell - \tfrac{1}{3} \gamma_k \gamma^\ell)\, \psi_\ell]$$

$$m_o{}^o = \frac{2}{3} \qquad m_o{}^k = 0$$

$$m_k{}^\ell = g_k{}^\ell \qquad m_k{}^o = 0$$

$$A_o{}^o = -\frac{2}{3}\gamma^o(\vec{\gamma}\cdot\vec{p}+m) \qquad A_k{}^\ell = -\gamma^o(\vec{\gamma}\cdot\vec{p}+m)\,g_k{}^\ell$$

$$A_k{}^o = -\frac{1}{3}p_k \qquad A_o{}^k = -\frac{1}{3}p^k$$

$$C^o = \frac{1}{3}(m+\frac{2}{3}\vec{\gamma}\cdot\vec{p}) \qquad C^k = \frac{1}{3}\gamma^o p^k \quad .$$

The Euler-Lagrange equations following from (2.14) are

$$m_\mu{}^\nu p_o \phi_\nu + A_\mu{}^\nu \phi_\nu + (C_\mu)^\dagger \lambda = 0 \tag{2.15}$$

$$C^\mu \phi_\mu = 0 \ . \tag{2.16}$$

Equation (2.16) reads

$$(m + \frac{2}{3}\vec{\gamma}\cdot\vec{p})\,\psi_o + \gamma^o p^k \phi_k = 0$$

which is the primary constraints expressing the spin 1/2 field ψ_o non-locally in terms of the spin 3/2 field.

The secondary constraint is obtained by multiplying (2.15) by $d^\mu \equiv C^\nu (m^{-1})_\nu{}^\mu$ and using (2.16). We obtain

$$d^\mu A_\mu{}^\nu \phi_\nu + (d\cdot c^\dagger)\lambda = 0 \qquad \ldots\ldots\ d\cdot c^\dagger \equiv d^\mu (C_\mu)^\dagger \quad .$$

This can be solved for λ to give

$$\lambda = \lambda_o - (d\cdot c^\dagger)^{-1} d^\mu A_\mu{}^\nu \phi_\nu$$

where

$$(d\cdot c^\dagger)\ \lambda_o = 0 \quad .$$

The field λ will not be given uniquely in terms of the fields ϕ_ν if a non-trivial λ_o exists. Such a λ_o does in

fact exist for the RS field in interaction with an electromagnetic field, leading to two inconsistencies in the theory which we shall next examine.

CHAPTER III

INCONSISTENCIES OF THE INTERACTING RS FIELD

A. Coupling to an External Electromagnetic Field

As mentioned previously, two basic inconsistencies arise when one attempts to take into account interactions in the RS theory. The first interaction we shall consider is minimal coupling to an electromagnetic field.

Throughout this work, we shall confine ourselves to an external electromagnetic field, the potentials A_μ taken to be classical entities.

i) Presence of Acausal Propagation

Velo and Zwanziger [1969a] noted that acausal propagation was possible in the classical RS field with minimal coupling in that propagation at speeds greater than that of light was possible. To do this, they showed that the RS equation was equivalent to a system of hyperbolic differential equations with initial conditions. By the method of characteristics, briefly outlined in Appendix II, one can compute the ray velocities. To illustrate the method we now show this calculation.

The RS equation (2.10) with minimal coupling is

$$(\gamma\cdot\pi - m)\,\psi_\mu - \frac{1}{3}\,(\gamma_\mu\pi^\nu + \pi_\mu\gamma^\nu)\,\psi_\nu + \frac{1}{3}\,\gamma_\mu(\gamma\cdot\pi + m)\gamma\cdot\psi = 0 \ . \qquad (3.1)$$

The maximum velocity of propagation will be given by the

slope of the characteristic surfaces. Causal propagation relies on the fact that no propagation can occur outside the light cone. We will now demonstrate, by calculating the normals to the characteristic surfaces of the system (3.1), that this condition is violated.

As discussed in the previous chapter, constraints are present in the RS equation of motion. Before calculating the normals, we must first eliminate these constraints in order to obtain the true equation of motion.

By contracting (3.1) with π^μ and γ^μ we obtain, respectively,

$$\frac{2}{3}\,\gamma\cdot\pi\pi\cdot\psi + \frac{ie}{6}\,\gamma\cdot F\cdot\gamma\gamma\cdot\psi - ie\gamma\cdot F\cdot\psi - m\pi\cdot\psi$$

$$+\frac{m}{3}\,\gamma\cdot\pi\gamma\cdot\psi = 0 \tag{3.2}$$

$$\pi\cdot\psi = -\frac{m}{2}\,\gamma\cdot\psi \tag{3.3}$$

where

$$\gamma\cdot F\cdot\psi = \gamma^\mu F_{\mu\nu}\psi^\nu$$

$$\gamma\cdot F\cdot\gamma = \gamma^\mu F_{\mu\nu}\gamma^\nu \quad .$$

Here, we have used the relation

$$[\pi_\mu,\pi_\nu] = ieF_{\mu\nu} \quad .$$

The primary constraint results from taking the zeroth component of (3.1) and using (3.3). It is

$$(m + \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\,\psi_o + \gamma^o\pi^k\phi_k = 0 \tag{3.4}$$

which is identical to (2.16), with $p_\mu \rightarrow \pi_\mu$.

The secondary constraints are obtained by substituting (3.3) into (3.2) and using the relations

$$(\gamma\cdot\pi)^2 = \pi^2 + \frac{ie}{2}\,\gamma\cdot F\cdot\gamma$$

$$\frac{ie}{2}\,\gamma\cdot F\cdot\gamma\gamma\cdot\psi - ie\gamma\cdot F\cdot\psi + ie\gamma_5\gamma\cdot\tilde{F}\cdot\psi = 0$$

where

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu}{}^{\sigma\lambda}F_{\sigma\lambda}$$

is the dual tensor to $F_{\mu\nu}$. The secondary constraint is

$$\gamma\cdot\psi = \frac{2ie}{3m^2}\,[\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi] \quad . \tag{3.5}$$

Using (3.3), we also obtain

$$\pi\cdot\psi = -\frac{ie}{3m}\,[\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi] \quad . \tag{3.6}$$

The true equation of motion results by substituting (3.5) and (3.6) into (3.1), which becomes

$$(\gamma\cdot\pi - m)\,\psi_\mu + \frac{ie}{9m}\,\gamma_\mu[\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi]$$

$$- \frac{2ie}{9m^2}\,[\pi_\mu - \gamma_\mu\gamma\cdot\pi - m][\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi] = 0 \quad . \tag{3.7}$$

The normals n_μ to the characteristic surfaces are computed by substituting n_μ for the highest derivative of the field components in (3.7). The characteristic determinant being a polynomial in the components of n_μ, we can, by Lorentz invariance, take $n_\mu = (n,0,0,0)$ and put the result into covariant form at the end. We obtain for the characteristic equation

$$D(n) = |n\gamma^{o} g_{\mu\nu} - \frac{2ie}{9m^2} (g_{o\mu} - \gamma_\mu\gamma_o)(\gamma_5\gamma_\sigma\tilde{F}^{\sigma}{}_{\nu} + 2\gamma_\sigma F^{\sigma}{}_{\nu})|$$

$$= n^{16}[1 - (\frac{2e}{3m^2})^2 \vec{B}^2]^4$$

$$= (n^2)^4[n^2 + (\frac{2e}{3m^2})^2 (n\cdot\tilde{F})^2]^4 = 0 \qquad (3.8)$$

where we have put the equation in covariant form in the last step. This is the equation which determines the normals to the characteristic surfaces which pass through each point.

It is now evident that (3.8) allows for propagation outside the lightcone in some Lorentz frame. This requires the existence of a timelike normal satisfying (3.8). For example, if we choose $n_\mu = (1,0,0,0)$, then we require

$$1 - (\frac{2e}{3m^2})^2 \vec{B}^2 = 0$$

which is satisfied in some Lorentz frame whenever a non-vanishing electromagnetic field exists. Indeed, the system loses its hyperbolicity if

$$1 - (\frac{2e}{3m^2})^2 \vec{B}^2 < 0$$

as there then exist normals with complex components which satisfy the characteristic equation. It might be hoped that the constraints eliminate these difficulties but, as Velo and Zwanziger [1969a] showed, the constraints are preserved in time and there exist disturbances, compatible with the constraints, which propagate outside the lightcone.

This demonstration of acausal propagation has been a great source of worry to theoretical physicists. Here is a system of equations which describes a free spin 3/2 particle, written in a Lorentz covariant form. However, when minimal coupling is introduced, one of the basic postulates of special relativity is violated. Shamaly and Capri [1972] attempted to correct the situation by the addition of general Pauli terms, but found the acausality was only enhanced. Inclusion of the internal electromagnetic field was also found to be unsuccessful in curing the problem (Capri and Shamaly [1974]). Evidently writing Lorentz covariant equations does not automatically guarantee that the postulates of special relativity will be satisfied.

ii) Existence of Indefinite Equal-Time Anticommutation Relations (ETACR)

A second inconsistency of the minimally coupled RS field was demonstrated by Johnson and Sudarshan [1961]. In their work, they calculated the equal-time anticommutation relations (ETACR) between the field components and found certain ones indefinite, whereas they should be positive definite by definition. To display the calculational technique, we will now derive these results.

We construct the generator of infinitesimal transformations of the field components

$$\delta\psi = i[G,\psi]$$

where

$$G = +i\int d^3y \, \frac{\partial \mathcal{L}}{\partial (p_o\psi_\mu)} \, \delta\psi_\mu \quad . \tag{3.9}$$

From the form of the Lagrangian (2.14), we find

$$G = -i\int d^3y [\phi^{k\dagger} \delta\phi_k + \frac{2}{3} \psi^{o\dagger} \delta\psi_o] \quad . \tag{3.10}$$

Only those field components which are independent are to be used in the expression for the generator. By substitution of the primary constraint (3.4),

$$\gamma^o\psi_o = -[m - \frac{2}{3} \vec{\gamma}\cdot\vec{\pi}]^{-1} \pi^k \phi_k$$

and its Hermitian conjugate into (3.10), we find

$$G = -i\int d^3y [\phi^{k\dagger} + \frac{2}{3} \phi^{m\dagger} \pi_m \bar{\Delta} \pi^k] \, \delta\phi_k \tag{3.11}$$

where

$$\bar{\Delta} = [m^2 - \frac{4}{9} (\vec{\gamma}\cdot\vec{\pi})^2]^{-1} \quad .$$

The ETACR may now be derived using (3.11). We find

$$\delta\phi_k(x) = i[G, \phi_k(x)]$$

$$= \int d^3y [\phi^{m\dagger} (g_m{}^\ell + \frac{2}{3} \pi_m \bar{\Delta} \pi^\ell) \delta\phi_\ell, \phi_k(x)] \ .$$

Using the identity

$$[AB, C] = A\{B, C\} - \{A, C\}B \ ,$$

we find

$$\delta\phi_k(x) = -\int d^3y \, \{\phi^{m\dagger}(y), \phi_k(x)\} (g_m{}^\ell + \frac{2}{3} \pi_m \bar{\Delta} \pi^\ell) \delta\phi_\ell(y) \tag{3.12}$$

assuming $\{\delta\phi_\ell(y), \phi_k(x)\} = 0 \implies \{\phi_k(x), \phi_\ell(y)\} = 0$.

Taking the ETACR of the form

$$\{\phi_k(x), \phi^{n\dagger}(y)\}\delta(x_o - y_o) = -P_k{}^i(g_i{}^j - \frac{2}{3}\pi_i\Delta\pi^j)P_j^n\delta(\vec{x}-\vec{y}) ,$$

where

$$P_k{}^\ell \equiv g_k{}^\ell - \frac{1}{3}\gamma_k\gamma^\ell ,$$

we find (3.12) is satisfied if

$$\bar{\Delta} - \Delta - \frac{2}{3}\Delta[\vec{\pi}^2 + \frac{1}{3}(\vec{\gamma}\cdot\vec{\pi})^2]\bar{\Delta} = 0 .$$

With $\bar{\Delta}$ given by (3.11), we therefore conclude that

$$\Delta = [m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B}]^{-1}$$

with

$$\vec{\sigma}\cdot\vec{B} = -\vec{\pi}^2 - (\vec{\gamma}\cdot\vec{\pi})^2 ,$$

$\vec{\sigma}$ being the vector formed from the antisymmetric tensor $\frac{i}{2}[\gamma_i, \gamma_j]$. Thus, we have arrived at the ETACR

$$\{\phi_k(x,)\phi^{\ell\dagger}(y)\} = -P_k{}^i(g_i{}^j - \frac{2}{3}\pi_i\Delta\pi^j)P_j^\ell\delta(\vec{x}-\vec{y})$$

$$\text{with} \quad \Delta = [m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B}]^{-1} . \tag{3.13}$$

Using the primary constraint (3.4), we can calculate other ETACR. Operating on (3.13) with π^k from the left, we obtain

$$\{\psi_o(x), \phi^{\ell\dagger}(y)\} = \gamma^o(m + \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta\pi^k P_k{}^\ell\delta(\vec{x}-\vec{y}) . \tag{3.14}$$

Operating on (3.14) by π^k from the left, we get

$$\{\psi_o(x),\psi_o^\dagger(y)\} = \frac{3}{2}[(m+\frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta(m-\frac{2}{3}\vec{\gamma}\cdot\vec{\pi}) - 1]\delta(\vec{x}-\vec{y}) \quad . \tag{3.15}$$

These ETACR are local, in spite of the non-local character of the constraint (3.4). Anticommutators involving the other spin 1/2 field $\vec{\gamma}\cdot\vec{\psi}$ will also be local, as $\vec{\gamma}\cdot\vec{\psi}$ is given locally by the secondary constraint (3.5).

We now show there is an inconsistency present in these relations. For simplicity, consider (3.15). Let $\chi(x)$ be an arbitrary spinor. Multiplying (3.15) by $\chi^\dagger(x)$ from the left and $\chi(y)$ from the right and integrating with respect to x and y yields the relation

$$\int\chi^\dagger(x)\{\psi_o(x),\psi_o^\dagger(y)\}\chi(y)\,dx\,dy =$$

$$= \frac{3}{2}\int\chi^\dagger(x)[(m+\frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta(m-\frac{2}{3}\vec{\gamma}\cdot\vec{\pi}) - 1]\chi(x)\,dx \quad . \tag{3.16}$$

Since the left side of (3.16) is positive definite, the right side must also be. However, if we define

$$\phi(x) = (m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\chi(x) \quad ,$$

then the right side of (3.16) becomes

$$\frac{3}{2}\int\phi^\dagger(x)[\Delta - \bar{\Delta}]\phi(x)\,dx$$

where

$$\bar{\Delta} = [m^2 + (\tfrac{2}{3})^2(\vec{\gamma}\cdot\vec{\pi})(\vec{\gamma}\cdot\vec{\pi})^\dagger]^{-1} \quad ,$$

which is a positive definite operator. However,

$$\Delta = [m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B}]^{-1}$$

is positive only if

$$m^2 > \frac{2e}{3} |\vec{B}| \quad .$$

Consequently, the right side of the relation (3.16) becomes negative for a sufficiently strong magnetic field, whereas it should always be positive definite by definition.

This again is a very troublesome result. The RS theory with minimal coupling is formally Lorentz covariant in the sense that the equations have the same form in all Lorentz frames. However, the work of Johnson and Sudarshan shows that inconsistencies arise only in those frames where the magnetic field is sufficiently strong; in other frames, no inconsistencies are present. Whereas the Velo-Zwanziger result violated the postulate of special relativity that no propagation can occur with a speed exceeding that of light, this result seems to violate the other postulate of special relativity that physical laws should be independent of the particular Lorentz frame in which they are derived. Inconsistencies, if they are to occur, should be present in all observer frames. This reinforces the claim that the postulates of special relativity and the writing of equations in a Lorentz covariant form are not necessarily compatible.

B. Coupling to a Spinor and a Scalar Field

A question that comes to mind is if the problems associated with minimal coupling are unique in the sense that acausality and the indefiniteness of the anti-

commutators is limited to this interaction.

All spin 3/2 particles known in nature so far decay into a scalar and a spin 1/2 particle. For example, the decay scheme of the N^* is

$$N^* \longrightarrow N\pi \quad .$$

It is of some physical interest then to explore the spin 3/2 field in interaction with a nucleon field and a pseudoscalar boson field. We shall see that the same sort of problems that persist in the case of electromagnetic coupling also occur here.

The Lagrangian for the RS field in interaction with a spinor field χ and a pseudoscalar field ρ is given by

$$\mathcal{L} = \bar{\psi}^{\mu}(\Gamma\cdot p - B)_{\mu}{}^{\nu}\psi_{\nu} + \bar{\chi}(\gamma\cdot p - M)\chi - \frac{1}{2}\,(p^{\nu}\rho p_{\nu}\rho + \mu^2\rho^2)$$

$$- f\,\bar{\psi}^{\mu}\theta_{\mu\nu}\chi p^{\nu}\rho \;-\; f(p^{\nu}\rho)\,\bar{\chi}\,\theta_{\nu\mu}\,\psi^{\mu}$$

where

$$\theta_{\mu\nu} = g_{\mu\nu} - h\gamma_{\mu}\gamma_{\nu} \quad .$$

The real constant h is in fact fixed by the requirement that constraints exist for the components ψ_{μ}.

Nath et al [1971] were the first to suggest this type of coupling with the aim of explaining the spin 3/2 contribution to πN scattering data. In this paper, it was concluded that the anticommutation relations did not suffer from the Johnson-Sudarshan difficulty. However, Hagen [1971] showed that these conclusions were in error. By quantizing the fields according to Schwinger's action

principle, he obtained the following ETACR:

$$\{\chi^{\dagger}(y),\chi(x)\} = m^2\,\Delta\,\delta(\vec{x}-\vec{y}) \tag{3.17}$$

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -P_k{}^i(g_i{}^j - \tfrac{2}{3}\,p_i\Delta p^j)P_j{}^{\ell}\,\delta(\vec{x}-\vec{y}) \tag{3.18}$$

where

$$\Delta = [m^2 - \tfrac{2}{3}\,f^2(\vec{p}\rho)^2]^{-1} \quad .$$

(For a complete listing, see Hagen [1971]).
These relations are indefinite for any $\vec{p}\rho \neq 0$, as may most easily be verified by examining (3.17). The similarity of (3.18) to that obtained for minimal coupling (see 3.13) is remarkable.

Singh [1973], and later Tait [1973], showed that, interpreted as a classical theory, there is acausal propagation present. As well as the usual lightcone characteristics, there are normals to the characteristic surfaces satisfying the equation

$$n^2 - \frac{2f^2}{3m^2}\,[(n^{\mu}p_{\mu}\rho)^2 - n^2(p^{\mu}\rho p_{\mu}\rho)] = 0 \ .$$

If it is required that $t = \text{constant}$ is a characteristic, then we obtain the condition

$$1 - \frac{2f^2}{3m^2}\,(\vec{p}\rho)^2 = 0$$

which is satisfied for a non-trivial ρ. This form is again very similar to the same condition obtained for the electromagnetic case,

$$1 - \left(\frac{2e}{3m^2}\right)^2\,\vec{B}^2 = 0 \ .$$

The results presented in this chapter suggest that the problems of causality and the indefiniteness of the anticommutation relations are related. The difficulties in both the c-number and q-number theories occurred in a similar set of Lorentz frames (i.e. where B and $\vec{p}\rho$ were sufficiently strong). As well, the problems were not restricted to one type of interaction. This leads to the hope that resolving one of the problems will lead to a solution of the other one.

CHAPTER IV

MODIFICATIONS OF THE MINIMALLY COUPLED RS FIELD

A. Addition of an External Scalar and a Spinor Field

As was discussed in the previous chapter, the RS field in interaction with a scalar and a spinor field as well as with minimal coupling suffers two inconsistencies. It would be interesting to see if a combination of the two interactions might lead to a consistent theory. An example of combining two acausal interactions that leads to a satisfactory theory has been given by Shamaly and Capri [1972] for the Takahashi-Palmer spin 1 Lagrangian. If such a cancellation of the undesirable effects is to occur here, then the scalar field ρ will have to become dependent on the electromagnetic field. Consequently, we will consider an external scalar field in the interaction.

i) Investigation of Causality

The Lagrangian is taken to be

$$\mathcal{L} = \bar{\psi}^{\mu}(\Gamma\cdot\pi - B)_{\mu}{}^{\nu}\psi_{\nu} + \bar{\chi}(\gamma\cdot\pi - M)\chi - \frac{1}{2}(p^{\nu}\rho p_{\nu}\rho + \mu^{2}\rho^{2})$$

$$- f\,\bar{\psi}^{\mu}\theta_{\mu\nu}\chi p^{\nu}\rho - f(p^{\nu}\rho)\,\bar{\chi}\,\theta_{\nu\mu}\,\psi^{\mu} \tag{4.1}$$

where

$$\theta_{\mu\nu} = g_{\mu\nu} - h\gamma_{\mu}\gamma_{\nu} \quad .$$

The equations of motion following from (4.1) are

$$(\gamma\cdot\pi - m)\psi_\mu - \frac{1}{3}(\gamma_\mu\pi^\nu + \pi_\mu\gamma^\nu)\psi_\nu + \frac{1}{3}\gamma_\mu(\gamma\cdot\pi + m)\gamma\cdot\psi$$

$$+ f\,\theta_{\mu\nu}\chi\, p^\nu\rho = 0 \qquad (4.2)$$

$$(\gamma\cdot\pi - M)\chi - f(p^\nu\rho)\,\theta_{\mu\nu}\psi^\mu = 0\ . \qquad (4.3)$$

Constraints are present in (4.2). Contracting this equation with γ^μ and π^μ yields, respectively,

$$\frac{2}{3}\pi\cdot\psi + \frac{m}{3}\gamma\cdot\psi + f\gamma^\mu\theta_{\mu\nu}\chi\, p^\nu\rho = 0 \qquad (4.4)$$

$$\frac{2}{3}\gamma\cdot\pi\pi\cdot\psi - \frac{ie}{3}[2\gamma\cdot F\cdot\psi + \gamma_5\gamma\cdot\tilde{F}\cdot\psi] - m\pi\cdot\psi + \frac{m}{3}\gamma\cdot\pi\gamma\cdot\psi$$

$$+ f(\pi^\mu\theta_{\mu\nu}\chi)\, p^\nu\rho + f\theta_{\mu\nu}\chi(p^\mu p^\nu\rho) = 0\ . \qquad (4.5)$$

The primary constraint results by taking the zeroth component of (4.2) and using (4.4). The result is

$$[\vec{\pi} + \frac{1}{3}\vec{\gamma}\cdot\vec{\pi}\vec{\gamma}]\cdot\vec{\psi} = f\gamma^k\theta_{k\nu}\chi p^\nu\rho + (m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\psi_o\ .$$

In order for constraints to emerge from this, we must have $\theta_{ko} = 0$. We thus conclude $h = 0$, and the primary constraint becomes

$$(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\psi_o + \pi^k\phi_k - f(\vec{\gamma}\cdot\vec{p}\rho)\chi = 0\ . \qquad (4.6)$$

Substituting (4.4) into (4.5) yields the secondary constraint

$$\frac{m^2}{2}\gamma\cdot\psi + \frac{3}{2m}f(\gamma\cdot p\rho)\chi - f(\pi^\mu\chi)\, p_\mu\rho + f(\gamma\cdot p\rho)(\gamma\cdot\pi\chi)$$

$$- \frac{ie}{3}[2\gamma\cdot F\cdot\psi + \gamma_5\gamma\cdot\tilde{F}\cdot\psi] = 0\ . \qquad (4.7)$$

By use of equation (4.3), we can also arrive at the result

$$m\pi\cdot\psi + f(\pi^{\mu}\chi)p_{\mu}\rho - Mf(\gamma\cdot p\rho)\chi - f^{2}(\gamma\cdot p\rho)(p_{\mu}\rho)\psi^{\mu}$$
$$+ \frac{ie}{3}\,[2\gamma\cdot F\cdot\psi + \gamma_5\gamma\cdot\tilde{F}\cdot\psi] = 0 \quad . \tag{4.8}$$

We are now ready to calculate the normals to the characteristic surfaces. To do this, we shall use the shock-wave formalism as developed by Madore and Tait [1973]. It is an equivalent method to that of finding the characteristic determinant as used by Velo and Zwanziger (see Appendix II). The shock-wave formalism exploits the fact that the first (and higher order) derivative of the solution of a first order system of hyperbolic equations suffers a discontinuity across a characteristic surface. Denoting this discontinuity by square brackets, we have

$$[p_{\mu}\psi_{\nu}] = n_{\mu}K_{\nu}$$

$$[p_{\mu}\chi] = n_{\mu}R$$

and

$$[\psi_{\nu}] = [\chi] = 0$$

where n_{μ} are the normals to the characteristic surface, with K_{ν} and R being continuously differentiable functions. Since we are interested in finding modes of propagation other than those on the lightcone, we shall assume in the following that $n^2 \neq 0$.

Taking the discontinuity of (4.3) leads to

$$\gamma\cdot nR = 0 \; .$$

Consequently, $R = 0$ and we can assume

$$[p_\mu p_\nu \chi] = n_\mu n_\nu T \quad .$$

Differentiating (4.3) and taking the discontinuity leads to

$$\gamma\cdot nT - fK^\nu p_\nu \rho = 0 \; , \tag{4.9}$$

while taking that of (4.8) gives the result

$$n\cdot K = 0 \; .$$

Taking the discontinuity of (4.2) gives

$$\gamma\cdot nK_\mu - \frac{1}{3}\, n_\mu \gamma\cdot K + \frac{1}{3}\, \gamma_\mu \gamma\cdot n\gamma\cdot K = 0 \; .$$

This enables us to write

$$K_\mu = [n_\mu - \gamma_\mu \gamma\cdot n]\psi \tag{4.10}$$

where

$$\psi = -\frac{(\gamma\cdot n)\gamma\cdot K}{3n^2} \quad .$$

Using this result in (4.9) leads to

$$T = f\gamma\cdot n[\gamma\cdot n(\gamma\cdot p\rho) - n\cdot p\rho]\psi/n^2 \quad . \tag{4.11}$$

Differentiating (4.7) and taking the discontinuity gives

$$\frac{3m^2}{2}\,\gamma\cdot n\psi + f[n\cdot p\rho - (\gamma\cdot p\rho)\gamma\cdot n]T + \frac{ie}{3}[2\gamma\cdot F\cdot K + \gamma_5 \gamma\cdot \tilde{F}\cdot K] = 0 \; .$$

Using (4.10) and (4.11) in this last expression leads to the conclusion that, for $\psi \neq 0$,

$$\{n^2 + \frac{2f^2}{3m^2}(n^2(p^\mu\rho p_\mu\rho) - (n^\mu p_\mu\rho)^2)\}^2 + (\frac{2e}{3m^2})^2 n^2(n\cdot\tilde{F})^2 = 0. \quad (4.12)$$

For a timelike normal $n = (1,0,0,0)$, we obtain the condition

$$\{1 - \frac{2f^2}{3m^2}(\vec{p}\rho)^2\}^2 - (\frac{2e}{3m^2})^2 \vec{B}^2 = 0 \ .$$

Thus, we have found that the propagation characteristics add for the two interactions. By examination of (4.12), we can conclude that there is no choice of the external field ρ that will cause the system to be causal. This effect of the propagation characteristics adding was also observed by Shamaly and Capri [1972], where various interactions between a spin 1/2 field and a spin 1 field were considered. There it was found that interactions that produced acausality individually could not be combined to give causal propagation, while a combination of causal interactions was still causal. However, when an acausal and a causal interaction were combined, the propagation was again acausal.

ii) Calculation of the ETACR

It would be interesting to see how the addition of the propagation characteristics in the c-number theory carries over into the quantized version. In this section we therefore calculate the various equal-time anti-commutation relations between the field components.

From the Lagrangian (4.1) we construct the generator

$$G = -i\int d^3y\{\phi^{k\dagger}\delta\phi_k + \frac{2}{3}\psi^{o\dagger}\delta\psi_o - \chi^\dagger\delta\chi\} \quad . \qquad (4.13)$$

The primary constraint (4.6) gives a relation between the variations of the field components.

$$(m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta\psi_o + \rho^k\delta\phi_k - f(\vec{\gamma}\cdot\vec{p}\rho)\delta\chi = 0\ . \qquad (4.14)$$

We shall handle this constraint by the method of Lagrange multiples (see, for example, Hagen [1971]). For a field component W(x), we write its variation as

$$\begin{aligned}\delta W(x) &= i[G,W(x)]\\ &= \int d^3y\{[\phi^{k\dagger}\delta\phi_k + \frac{2}{3}\,\psi^{o\dagger}\delta\psi_o - \chi^\dagger\delta\chi, W(x)]\\ &+ \Lambda(x,y)[(m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta\psi_o + \pi^k\delta\phi_k - f(\vec{\gamma}\cdot\vec{p}\rho)\delta\chi]\} \qquad (4.15)\end{aligned}$$

and treat all components as independent. The Lagrange multiplier field is then determined so as the primary constraint (4.6) is satisfied.

For $W(x) = \chi(x)$, (4.15) gives

$$\{\phi^{k\dagger}(y),\chi(x)\} = \Lambda(x,y)\pi^m(g_m{}^k - \frac{1}{3}\,\gamma_m\gamma^k)$$

$$\{\psi^{o\dagger}(y),\chi(x)\} = \Lambda(x,y)(m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\,\gamma^o$$

$$\{\chi^\dagger(y),\chi(x)\} = \delta(\vec{x}-\vec{y}) + f\Lambda(x,y)(\vec{\gamma}\cdot\vec{p}\rho)\quad .$$

We find (4.6) is satisfied if

$$\Lambda(x,y) = -\,\frac{2}{3}\,f(\vec{\gamma}\cdot\vec{p}\rho)\Delta\delta(\vec{x}-\vec{y})$$

where

$$\Delta = [m^2 + \frac{2e}{3}\,\vec{\sigma}\cdot\vec{B} - \frac{2}{3}\,f^2(\vec{p}\rho)^2]^{-1}\ .$$

The ETACR then become

$$\{\phi^{k\dagger}(y),\chi(x)\} = -\frac{2}{3}\, f(\vec{\gamma}\cdot\vec{p}\rho)\Delta\pi^m (g_m{}^k - \frac{1}{3}\,\gamma_m\gamma^k)\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\chi(x)\} = -\, f(\vec{\gamma}\cdot\vec{p}\rho)\Delta(m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\,\gamma^o\,\delta(\vec{x}-\vec{y})$$

$$\{\chi^\dagger(y),\chi(x)\} = (m^2 + \frac{2e}{3}\,\vec{\sigma}\cdot\vec{B})\Delta\delta(\vec{x}-\vec{y}) \qquad .$$

For $W(x) = \psi_o(x)$, we find

$$\{\phi^{k\dagger}(y),\psi_o(x)\} = \gamma^o(m + \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\Delta\pi^m(g_m{}^k - \frac{1}{3}\,\gamma_m\gamma^k)\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\psi_o(x)\} = \frac{3}{2}\,[(m + \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\Delta(m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi}) - 1]\,\delta(\vec{x}-\vec{y})$$

$$\{\chi^\dagger(y),\,\psi_o(x)\} = f\gamma^o(m + \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\Delta(\vec{\gamma}\cdot\vec{p}\rho)\delta(\vec{x}-\vec{y})$$

while for $W(x) = \phi_k(x)$, we obtain

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -\,P_k{}^n(g_n{}^m - \frac{2}{3}\,\pi_n\Delta\pi^m)P_m{}^\ell\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\phi_k(x)\} = (g_k{}^\ell - \frac{1}{3}\,\gamma_k\gamma^\ell)\pi_\ell\Delta(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta(\vec{x}-\vec{y})$$

$$\{\chi^\dagger(y),\,\phi_k(x)\} = \frac{2}{3}\,f(g_k{}^\ell - \frac{1}{3}\,\gamma_k\gamma^\ell)\pi_\ell\Delta(\vec{\gamma}\cdot\vec{p}\rho)\delta(\vec{x}-\vec{y})$$

where

$$P_k{}^\ell = g_k{}^\ell - \frac{1}{3}\,\gamma_k\gamma^\ell \qquad .$$

We see that this set of relations is mutually consistent; for example, the result for $\{\chi^\dagger(y),\psi_o(x)\}$ is compatible with that for $\{\psi^{o\dagger}(y),\chi(x)\}$. The ETACR are then

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -P_k{}^m(g_m{}^n - \frac{2}{3}\,\pi_m\Delta\pi^n)P_n{}^\ell\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\phi_k(x)\} = P_k{}^\ell\pi_\ell\Delta(m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta(\vec{x}-\vec{y})$$

$$\{\chi^\dagger(y),\phi_k(x)\} = \frac{2}{3}\, f P_k{}^\ell \pi_\ell \Delta(\vec{\gamma}\cdot\vec{p}\rho)\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\psi_o(x)\} = \frac{3}{2}\,[(m+\frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\Delta(m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi}) - 1]\delta(\vec{x}-\vec{y})$$

$$\{\chi^\dagger(y),\psi_o(x)\} = f\gamma^o(m + \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\Delta(\vec{\gamma}\cdot\vec{p}\rho)\delta(\vec{x}-\vec{y})$$

$$\{\chi^\dagger(y),\chi(x)\} = (m^2 + \frac{2e}{3}\,\vec{\sigma}\cdot\vec{B})\Delta\delta(\vec{x}-\vec{y}) \qquad (4.16)$$

where

$$P_k{}^\ell = g_k{}^\ell - \frac{1}{3}\,\gamma_k\gamma^\ell$$

$$\Delta = [m^2 + \frac{2e}{3}\,\vec{\sigma}\cdot\vec{B} - \frac{2}{3}\,f^2(\vec{p}\rho)^2]^{-1} \quad .$$

We see that there will still be an indefiniteness problem by considering, most conveniently, the anticommutator between $\chi^\dagger(y)$ and $\chi(x)$. We also find that the indefiniteness associated with the two interactions individually in a sense added for the combined interaction, but in a fashion that still was indefinite. For example, consider the relation between $\phi^{\ell\dagger}(y)$ and $\phi_k(x)$. This can be obtained from the same relation found in the case of the electromagnetic interaction alone (see 3.13) by the replacement

$$\Delta = [m^2 + \frac{2e}{3}\,\vec{\sigma}\cdot\vec{B}]^{-1} \longrightarrow \Delta = [m^2 + \frac{2e}{3}\,\vec{\sigma}\cdot\vec{B} - \frac{2}{3}\,f^2(\vec{p}\rho)^2]^{-1} .$$

On the other hand, the same relation can be obtained from the result found in the case of the scalar-spinor interaction alone (see 3.18) by the replacement

$$p_k \longrightarrow \pi_k$$

$$\Delta = [m^2 - \frac{2}{3} f^2 (\vec{p}\rho)^2]^{-1} \longrightarrow \Delta = [m^2 + \frac{2e}{3} \vec{\sigma}\cdot\vec{B} - \frac{2}{3} f^2 (\vec{p}\rho)^2]^{-1} .$$

Thus, the addition of the propagation characteristics in the c-number theory seems to carry over into addition of the indefiniteness of the anticommutators in the q-number theory. This again suggests the two problems are closely connected.

B. Addition of a Spinor Field by Scalar Coupling

To attempt to eliminate the difficulties inherent in the charged spin 3/2 theory, we might compare this theory to that of spin 1/2. The latter is free of the inconsistencies considered here in both the c-number and q-number formulations. The essential difference between the RS and spin 1/2 fields is that the former contains auxiliary fields (of spin 1/2) which must be eliminated so as to be left with a pure spin 3/2 state. It may then be worthwhile to see if the addition of a spin 1/2 field $\psi(x)$ to the RS field might, by a suitable choice of the coupling constants, be made to somehow eliminate the inconsistencies present in the original system. For the present we concentrate on the minimally coupled RS field.

i) Investigation of Causality

We first try scalar and pseudoscalar coupling. The Lagrangian is

$$\mathcal{L} = \bar{\psi}^{\mu}(\Gamma\cdot\pi - B)_{\mu}{}^{\nu}\psi_{\nu} + \bar{\psi}(\gamma\cdot\pi - \lambda m)\psi + \bar{\psi}\cdot\gamma(a+b\gamma_5)\psi$$

$$+ \bar{\psi}(a+b\gamma_5)\gamma\cdot\psi \qquad (4.17)$$

where

m is the mass of the spin 3/2 particle

λm is the mass of the spin 1/2 particle .

From the Lagrangian (4.17), we obtain the equations of motion

$$(\gamma\cdot\pi - m)\psi_{\mu} - \frac{1}{3}(\gamma_{\mu}\pi^{\nu} + \pi_{\mu}\gamma^{\nu})\psi_{\nu} + \frac{1}{3}\gamma_{\mu}(\gamma\cdot\pi + m)\gamma\cdot\psi$$

$$- \gamma_{\mu}(a+b\gamma_5)\psi = 0 \qquad (4.18)$$

$$(\gamma\cdot\pi - \lambda m)\psi + (a+b\gamma_5)\gamma\cdot\psi = 0 \ . \qquad (4.19)$$

We must eliminate the constraints present in (4.18) before finding the normals to the characteristic surfaces. Contracting (4.18) with γ^{μ} and π^{μ} gives, respectively,

$$\pi\cdot\psi = -\frac{m}{2}\gamma\cdot\psi + 6(a+b\gamma_5)\psi \qquad (4.20)$$

$$\frac{2}{3}\gamma\cdot\pi\pi\cdot\psi - ie\gamma\cdot F\cdot\psi + \frac{ie}{6}\gamma\cdot F\cdot\gamma\gamma\cdot\psi - m\pi\cdot\psi + \frac{m}{3}\gamma\cdot\pi\gamma\cdot\psi$$

$$- \gamma\cdot\pi(a+b\gamma_5)\psi = 0 \ . \qquad (4.21)$$

The primary constraint is obtained by taking the zeroth component of (4.18) and using (4.20). The result is

$$[\vec{\pi} + \frac{1}{3}\vec{\gamma}\cdot\vec{\pi}\vec{\gamma}]\cdot\vec{\psi} + 3(a+b\gamma_5)\psi = (m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^{o}\psi_{o} \ . \qquad (4.22)$$

To find the secondary constraint, substitute (4.20) into (4.21) and use (4.14). The result is

$$\gamma\cdot\psi = [1 - \frac{6g^2}{m^2}]^{-1}\{\frac{2ie}{3m^2}(\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi) + \frac{12}{m}(a+b\gamma_5)\psi$$

$$- \frac{6}{m}\lambda(a-b\gamma_5)\psi\} \qquad (4.23)$$

where $g^2 = a^2 + b^2$ and $6g^2 \neq m^2$.

Using (4.20), we also obtain the relation

$$\pi\cdot\psi = -[1 - \frac{6g^2}{m^2}]^{-1}\{\frac{ie}{3m}(\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi)$$

$$+6(a+b\gamma_5)\psi - 3\lambda(a-b\gamma_5)\psi\} + 6(a+b\gamma_5)\psi \quad . \qquad (4.24)$$

To find the true equation of motion, substitute (4.23) and (4.24) into (4.18). Keeping terms with derivatives only, this becomes

$$\gamma\cdot\pi\psi_\mu - \frac{1}{3}[1-\frac{6g^2}{m^2}]^{-1}[\pi_\mu - \gamma_\mu\gamma\cdot\pi]\{\frac{2ie}{3m^2}(\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi)$$

$$+\frac{12}{m}(a+b\gamma_5)\psi - \frac{6}{m}\lambda(a-b\gamma_5)\psi\} = 0 \ . \qquad (4.25)$$

To calculate the characteristic determinant, substitute $n_\mu = (n,0,0,0)$ for p_μ in (4.25) and (4.19). The resulting equations in terms of n are

$$n\gamma^o\psi_\mu - \frac{n}{3}[1-\frac{6g^2}{m^2}]^{-1}[g_{o\mu} - \gamma_\mu\gamma_o]\{\frac{2ie}{3m^2}(\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi)$$

$$+\frac{12}{m}(a+b\gamma_5)\psi - \frac{6}{m}\lambda(a-b\gamma_5)\psi\} = 0 \qquad (4.26)$$

$$n\gamma^o\psi = 0 \quad . \qquad (4.27)$$

Taking the determinant of the coefficients of the field components in (4.26) and (4.27), we obtain for the characteristic equation

$$D(n) = n^{20}\left[1 - \left(\frac{2e}{3m^2}\right)^2 \vec{B}^2/\left(1 - \frac{6g^2}{m^2}\right)^2\right]^4$$

$$= (n^2)^6\left[n^2 + \left(\frac{2e}{3m^2}\right)^2 (n\cdot\tilde{F})^2/\left(1 - \frac{6g^2}{m^2}\right)^2\right]^4$$

$$= 0$$

where we have put the equation into covariant form in the last step. Thus, there still exists acausal propagation, as there are timelike normals which satisfy this equation for a suitable Lorentz frame. The condition that the system remains hyperbolic is

$$1 - \left(\frac{2e}{3m^2}\right)^2 \vec{B}^2/\left(1 - \frac{6g^2}{m^2}\right)^2 > 0$$

while that for pure minimal coupling was

$$1 - \left(\frac{2e}{3m^2}\right)^2 \vec{B}^2 > 0 \; .$$

We now see the effect of the addition of the spin 1/2 field. For $m^2 > 6g^2$, the upper limit on the magnitude of the magnetic field needed to keep the system hyperbolic is decreased, compared to the case $g^2 = 0$. When $m^2 < 6g^2$, this limit is increased. However, for any $6g^2 \neq m^2$, there exists Lorentz frames where causality is violated and the system loses its hyperbolicity.

ii) Calculation of the ETACR

It would be worthwhile to see how the addition of the spin 1/2 field affects the anticommutation relations. From the Lagrangian (4.17), we can construct the generator

$$G = -i\int d^3y\{\phi^{k\dagger}\delta\phi_k + \frac{2}{3}\psi^{o\dagger}\delta\psi_o - \psi^\dagger\delta\psi\}$$

with the field components subject to the primary constraint (4.22),

$$(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta\psi_o + \pi^k\delta\phi_k - 3(a + b\gamma_5)\delta\psi = 0 \ .$$

We shall again take into account this constraint in the expression for the generator by the method of Lagrange multipliers. We write for the variation of a field component W(x),

$$\begin{aligned}\delta W(x) &= i[G,W(x)]\\ &= \int d^3y\{[\phi^{k\dagger}\delta\phi_k + \frac{2}{3}\psi^{o\dagger}\delta\psi_o - \psi^\dagger\delta\psi, W(x)]\\ &+ \Lambda(x,y)[(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta\psi_o + \pi^k\delta\phi_k - 3(a + b\gamma_5)\delta\psi]\} \ .\end{aligned}$$

For $W(x) = \psi(x)$, we obtain

$$\{\phi^{k\dagger}(y),\psi(x)\} = \Lambda(x,y)\pi^m(g_m{}^k - \frac{1}{3}\gamma_m\gamma^k)$$

$$\{\psi^{o\dagger}(y),\psi(x)\} = \frac{3}{2}\Lambda(x,y)(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o$$

$$\{\psi^\dagger(y), \psi(x)\} = \delta(\vec{x}-\vec{y}) + 3\Lambda(x,y)(a + b\gamma_5).$$

In order that the primary constraint (4.22) is satisfied, we must have

$$\Lambda(x,y) = 2(a - b\gamma_5)\Delta\delta(\vec{x}-\vec{y})$$

where

$$\Delta = [m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B} - 6g^2]^{-1}$$

and

$$g^2 = a^2 + b^2 .$$

We then arrive at the ETACR

$$\{\phi^{k\dagger}(y),\psi(x)\} = 2(a - b\gamma_5)\Delta\pi^m(g_m^{\ k} - \frac{1}{3}\gamma_m\gamma^k)\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\psi(x)\} = 3(a - b\gamma_5)\Delta(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta(\vec{x}-\vec{y})$$

$$\{\psi^{\dagger}(y),\psi(x)\} = (m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B})\Delta\delta(\vec{x}-\vec{y}) .$$

For $W(x) = \psi_o(x)$, we obtain

$$\{\phi^{k\dagger}(y),\psi_o(x)\} = \gamma^o(m + \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta\pi^m(g_m^{\ k} - \frac{1}{3}\gamma_m\gamma^k)\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\psi_o(x)\} = \frac{3}{2}[(m + \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi}) - 1]\delta(\vec{x}-\vec{y})$$

$$\{\psi^{\dagger}(y),\psi_o(x)\} = 3\gamma^o(m + \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta(a + b\gamma_5)\delta(\vec{x}-\vec{y}) ,$$

while for $W(x) = \phi_k(x)$, we arrive at

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -P_k^{\ m}(g_m^{\ n} - \frac{2}{3}\pi_m\Delta\pi^n)P_n^{\ \ell}\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\phi_k(x)\} = (g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\pi_\ell\Delta(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\delta(\vec{x}-\vec{y})$$

$$\{\psi^{\dagger}(y),\phi_k(x)\} = 2(g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\pi_\ell\Delta(a + b\gamma_5)\delta(\vec{x}-\vec{y}) .$$

Again, these relations are mutually consistent, as can be seen by examining, for example, $\{\psi^{\dagger}(y),\phi_k(x)\}$ and

$\{\phi^{k\dagger}(y),\psi(x)\}$. The complete set of ETACR are then

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -P_k{}^m(g_m{}^n - \frac{2}{3}\pi_m\Delta\pi^n)\, P_n{}^\ell\, \delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\phi_k(x)\} = P_k{}^\ell \pi_\ell\, \Delta\, (m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o \delta(\vec{x}-\vec{y})$$

$$\{\psi^\dagger(y),\phi_k(x)\} = 2P_k{}^\ell \pi_\ell \Delta (a + b\gamma_5)\,\delta(\vec{x}-\vec{y})$$

$$\{\psi^\dagger(y),\psi_o(x)\} = 3\gamma^o(m + \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta(a + b\gamma_5)\,\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\psi_o(x)\} = \frac{3}{2}\,[(m + \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\Delta(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi}) - 1]\,\delta(\vec{x}-\vec{y})$$

$$\{\psi^\dagger(y),\psi(x)\} = (m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B})\Delta\delta(\vec{x}-\vec{y}) \qquad (4.28)$$

where

$$P_k{}^\ell = g_k{}^\ell - \frac{1}{3}\gamma_k\gamma^\ell$$

$$\Delta = [m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B} - 6g^2]^{-1} \quad .$$

Again, these relations are indefinite, as can be seen by examining say, $\{\psi^{o\dagger}(y),\psi_o(x)\}$ along the same lines as used before in the case of minimal coupling (see 3.15 and the subsequent discussion). However, the condition for the anticommutator to be positive definite has been altered by inclusion of the interaction with the spin 1/2 field. With only minimal coupling, this condition was

$$\frac{2e}{3}\,|\vec{B}| < m^2$$

while the same condition with the spin 1/2 field included is

$$\frac{2e}{3}\,|\vec{B}| < |m^2 - 6g^2| \quad .$$

Thus, for $m^2 > 6g^2$, the upper limit on the magnitude of the magnetic field strength needed to have the anticommutator positive is decreased, compared to the case $g^2 = 0$. For $m^2 < 6g^2$, the same limit is increased. However, there still exist Lorentz frames in which the anticommutator is negative for all $6g^2 \neq m^2$. These effects are similar to those found for the conditions of hyperbolicity in the c-number theory, again suggesting that the problems of causality and the indefiniteness of the anticommutators are related.

CHAPTER V

THE BHABHA-GUPTA FIELD

Since the addition of a spin 1/2 field to the RS field by scalar coupling was seen to have the same problems inherent in the minimally coupled RS equation, it would seem natural to try the same two fields with derivative coupling. We would expect that this type of coupling has more of an impact on the propagation characteristics and anticommutation relations than did scalar coupling, where only the constraints were changed. The presence of derivative coupling will also alter the principal part of the equations of motion (i.e. introduce more terms involving derivatives) as well as change the form of the generator.

We start from the Lagrangian

$$\mathcal{L} = \bar{\psi}^{\mu}(\Gamma\cdot p - B)_{\mu}{}^{\nu}\psi_{\nu} + d\bar{\psi}(\gamma\cdot p - \lambda m)\psi - \bar{\psi}(a + b\gamma_5)p\cdot\psi$$
$$- \bar{\psi}\cdot p(a + b\gamma_5)\psi \quad . \tag{5.1}$$

This system is in fact equivalent to one proposed by Bhabha [1952] for a particle of two mass states (with $b = 0$). Bhabha's original equation involved a 20 component wave function and was written in the form

$$(\alpha\cdot p - m)\psi = 0 \ .$$

One state, of mass m, had spin 3/2, while the other state, of mass λm, had spin 1/2. The parameter λ is an arbitrary

real number, while d, also real, is a constant to be discussed shortly. By computing the matrix α^0, Gupta [1954] showed that the system (5.1) was equivalent to Bhabha's original field (for $b = 0$). For the rest of this work, we shall denote the system (5.1) as the Bhabha-Gupta (BG) field.

A. The Minimally Coupled BG Field

i) Investigation of Causality

Introducing minimal coupling in the Langrangian (5.1), we find for the equations of motion

$$(\gamma\cdot\pi - m)\psi_\mu - \frac{1}{3}(\gamma_\mu\pi^\nu + \pi_\mu\gamma^\nu)\psi_\nu + \frac{1}{3}\gamma_\mu(\gamma\cdot\pi + m)\gamma\cdot\psi$$

$$+ (a + b\gamma_5)\pi_\nu\psi = 0 \tag{5.2}$$

$$d(\gamma\cdot\pi - \lambda m)\psi - (a + b\gamma_5)\pi\cdot\psi = 0 \quad . \tag{5.3}$$

To eliminate the constraints, contract (5.2) with γ^μ and π^μ respectively to obtain

$$\pi\cdot\psi = -\frac{m}{2}\gamma\cdot\psi - \frac{3}{2}(a - b\gamma_5)\gamma\cdot\pi\psi \tag{5.4}$$

$$\frac{2}{3}\gamma\cdot\pi\pi\cdot\psi - ie\gamma\cdot F\cdot\psi + \frac{ie}{6}\gamma\cdot F\cdot\gamma\gamma\cdot\psi - m\pi\cdot\psi + \frac{m}{3}\gamma\cdot\pi\gamma\cdot\psi$$

$$+ (a + b\gamma_5)\pi^2\psi = 0 \quad . \tag{5.5}$$

The primary constraint results from taking the zeroth component of (5.2) and using (5.4). It is

$$[\vec{\pi} + \frac{1}{3}\,\vec{\gamma}\cdot\vec{\pi}\vec{\gamma}]\cdot\vec{\psi} + \vec{\gamma}\cdot\vec{\pi}(a + b\gamma_5)\psi = (m - \frac{2}{3}\,\vec{\gamma}\cdot\vec{\pi})\gamma^o\psi_o \quad . \tag{5.6}$$

To find the secondary constraints, substitute (5.4) into (5.5) and use (5.3). The result is

$$\gamma\cdot\psi = \frac{2ie}{3m^2}\,(1 + \frac{3g^2}{2d})[\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi + \frac{3}{2}\,(a + b\gamma_5)\gamma\cdot F\cdot\gamma\psi]$$

$$- 3\lambda(a - b\gamma_5)\psi \tag{5.7}$$

where $g^2 = a^2 + b^2$.

Using (5.4) and (5.3), we also obtain the relation

$$\pi\cdot\psi = -\frac{ie}{3m}\,[\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi + \frac{3}{2}\,(a + b\gamma_5)\gamma\cdot F\cdot\gamma\psi] \quad . \tag{5.8}$$

The true equation of motion results from substituting (5.7) and (5.8) into (5.2). To calculate the characteristic determinant, substitute $n_\mu = (n,0,0,0)$ in for p_μ in this equation and in (5.3). The resulting equations in terms of n are

$$n\gamma^o\psi_\mu - \frac{2ie}{9m^2}n(1 + \frac{3g^2}{2d})(g_{o\mu} - \gamma_\mu\gamma_o)\ [\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi$$

$$+ \frac{3}{2}\,(a + b\gamma_5)\gamma\cdot F\cdot\gamma\psi] + n\,g_{o\mu}[(a + b\gamma_5) + \lambda(a - b\gamma_5)]\psi$$

$$- n\,\lambda\gamma_\mu\gamma_o(a - b\gamma_5)\psi = 0$$

$$n\,d\,\gamma^o\psi - n(a + b\gamma_5)\,\psi_o = 0 \quad .$$

The characteristic determinant of this system of equations is

$$D(n) = n^{20}[1 - (\frac{2e}{3m^2})^2 B^2 (1 + \frac{3g^2}{2d})^2]^4$$

$$= (n^2)^6 [n^2 + (\frac{2e}{3m^2})^2 (1 + \frac{3g^2}{2d})^2 (n\cdot F)^2]^4 = 0 \qquad (5.9)$$

where we have cast the equation into covariant form in the last step.

Before proceeding with an analysis of the propagation characteristics, let us see what the significance of the parameter 'd' is. To do this, we calculate the four-vector current density from the free Lagrangian (5.1). We obtain

$$J^\sigma = \frac{\partial \mathcal{L}}{\partial (p_\sigma \psi_\mu)} \psi_\mu + \frac{\partial \mathcal{L}}{\partial (p_\sigma \psi)} \psi$$

$$= -\bar{\psi}^\mu \gamma^\sigma \psi_\mu + \frac{1}{3} \bar{\psi}^\sigma \gamma\cdot\psi + \frac{1}{3} \bar{\psi}\cdot\gamma\psi^\sigma - \frac{1}{3} \bar{\psi}\cdot\gamma\gamma^\sigma\gamma\cdot\psi$$

$$+ d\bar{\psi}\gamma^\sigma\psi - \bar{\psi}(a + b\gamma_5)\psi^\sigma - \bar{\psi}^\sigma(a + b\gamma_5)\psi \quad .$$

The charge density J^o has to be positive definite if a consistent quantization on a positive definite metric is to be achieved. From $p\cdot\psi = 0$ from equation (5.8), we conclude $\psi^o = 0$ in the rest frame. Calculating J^o in the rest frame, we find

$$J^o_{rest} = [\vec{\psi} + \frac{1}{3} \vec{\gamma}\vec{\gamma}\cdot\vec{\psi}]^\dagger \cdot [\vec{\psi} + \frac{1}{3} \vec{\gamma}\vec{\gamma}\cdot\vec{\psi}] + d\,\psi^\dagger\psi \quad .$$

Thus, if $d > 0$, the charge density is positive definite. However, from (5.9), we see that if $d < 0$, we can have causal propagation by choosing $g^2 = |\frac{2}{3}d|$, whereby the characteristic equation becomes

$$(n^2)^{10} = 0 \quad .$$

The situation is a trade-off: we can either have acausal propagation and a positive definite charge density, or else causal propagation with an indefinite charge density. It would be interesting to see if the indefiniteness of the anticommutation relations is eliminated as well by choosing $d < 0$. However, we should be prepared from the start to expect an indefinite metric is needed for quantization, due to the indefiniteness of the total free charge density.

ii) Calculation of the ETACR

From the Lagrangian (5.1), we can construct the generator of infinitesimal transformations of the field components,

$$G = -i\int d^3y\{[\tfrac{2}{3}\bar{\psi}^o + \psi^\dagger(a - b\gamma_5)]\gamma^o\delta\psi_o + \phi^{k\dagger}\delta\phi_k + [\bar{\psi}^o(a + b\gamma_5) - d\psi^\dagger]\delta\psi\} \quad . \tag{5.10}$$

From the primary constant (5.6),

$$(m - \tfrac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\psi_o + \pi^k\phi_k - \vec{\gamma}\cdot\vec{\pi}(a + b\gamma_5)\psi = 0$$

and its Hermitian conjugate, we may eliminate all terms containing ψ^o in the expression (5.10). The result is

$$G = -i\int d^3y\{[\phi^{k\dagger} - m\psi^\dagger(a - b\gamma_5)\bar{\Delta}\pi^k + \tfrac{2}{3}\phi^{\ell\dagger}\pi_\ell\bar{\Delta}\pi^k]\delta\phi_k - [d\psi^\dagger + m\phi^{k\dagger}\pi_k\bar{\Delta}(a + b\gamma_5) - \tfrac{2}{3}g^2\psi^\dagger\bar{\Delta}]\delta\psi\} \tag{5.11}$$

where

$$\bar{\Delta} = [m^2 - \frac{4}{9}(\vec{\gamma}\cdot\vec{\pi})^2]^{-1}$$

$$g^2 = a^2 + b^2 \ .$$

We find we can diagonalize the generator (5.11) by casting it into the form

$$G = -i\int d^3y[\chi^\dagger E\delta\chi + \phi^{k\dagger}A_k{}^\ell\delta\phi_\ell] \tag{5.12}$$

where

$$\chi = \psi - m(a - b\gamma_5)E^{-1}\bar{\Delta}\pi^k\phi_k$$

$$E = -d - \frac{3}{2}g^2 + \frac{3}{2}m^2g^2\bar{\Delta}$$

$$A_k{}^\ell = g_k{}^\ell - \frac{2}{3}\pi_k(\frac{3}{2}g^2 + d)E^{-1}\bar{\Delta}\pi^\ell \ .$$

We can now easily derive the ETACR. For example, by the relation

$$\delta\chi(x) = i[G,\chi(x)]$$

we arrive at

$$\{\phi^{k\dagger}(y),\chi(x)\} = 0$$

and

$$\{\chi^\dagger(y),\chi(x)\} = -E^{-1}\delta(\vec{x}-\vec{y}) \ . \tag{5.13}$$

Similarly, by use of

$$\delta\phi_k(x) = i[G,\phi_k(x)]$$

we find

$$\{\chi^\dagger(y),\phi_k(x)\} = 0$$

and

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -P_k{}^m(g_m{}^n - \tfrac{2}{3}\pi_m\Delta\pi^n)P_n{}^\ell\,\delta(\vec{x}-\vec{y}) \qquad (5.14)$$

where

$$P_k{}^\ell = g_k{}^\ell - \tfrac{1}{3}\gamma_k\gamma^\ell$$

$$\Delta = [m^2/(1+\frac{3g^2}{2d}) + \frac{2e}{3}\vec{\sigma}\cdot\vec{B}]^{-1} \; .$$

For a positive charge density, $d > 0$. Choosing $d = 1$, we have, from (5.13),

$$\{\chi^\dagger(y),\chi(x)\} = [1+\tfrac{3}{2}g^2 - \tfrac{3}{2}m^2g^2\bar{\Delta}]^{-1}\delta(\vec{x}-\vec{y})$$

and by (5.14),

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -P_k\,(g_m{}^n - \tfrac{2}{3}\pi_m\Delta\pi^n)P_n{}^\ell\,\delta(\vec{x}-\vec{y})$$

where

$$\Delta = [m^2/(1+\tfrac{3}{2}g^2) + \frac{2e}{3}\vec{\sigma}\cdot\vec{B}]^{-1} \quad .$$

These are the same results as obtained by Johnson and Sudarshan [1961]. There, the latter anticommutator was shown to be indefinite, and the BG theory with positive charge density thus encounters the same difficulties as the RS field does.

We now recall that causal propagation is possible if the choice $d = -\frac{3}{2}g^2$ is made. We are now in a position to discover if the indefinite anticommutator problem is also eliminated by this choice. Without loss of generality, we take $d = -1$, since the field ψ can be renormalized to meet

this condition. Imposing $g^2 = a^2 + b^2 = \frac{2}{3}$, we find, by (5.13),

$$\{\chi^\dagger(y),\chi(x)\} = -\frac{1}{m^2}\,[m^2 + (\tfrac{2}{3})^2 (\vec{\gamma}\cdot\vec{\pi})(\vec{\gamma}\cdot\vec{\pi})^\dagger]\,\delta(\vec{x}-\vec{y}) \tag{5.15}$$

and by (5.14),

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -(g_k^{\ell} - \tfrac{1}{3}\gamma_k\gamma^{\ell})\,\delta(\vec{x}-\vec{y}) \ . \tag{5.16}$$

We see that the right side of (5.15) is negative definite, while the right side of (5.16) is positive definite. As expected, an indefinite metric is needed upon quantization. Successful indefinite metric theories require that all observables be associated with that part of the Hilbert space where the metric is positive definite. Such theories have been constructed for various interacting field problems, particularly QED (for a general discussion of indefinite metric theories, see Sudarshan [1968], Nakanishi [1972], Sudarshan [1972]). The situation presented here is perhaps more favourable than one involving indefinite anticommutators, where inconsistencies arose only in those Lorentz frames where the magnetic field was sufficiently strong. The anticommutators (5.14) and (5.15) do not possess such definiteness. However, before exploring the possibility of incorporating a consistent indefinite metric theory in the BG field, it would be worthwhile to calculate the ETACR between all the field components to ensure that there is no indefiniteness present.

B. The ETACR in Terms of the Independent Field Components

i) Determination of the Independent Fields

Before proceeding with the calculation of the complete set of anticommutation relations, let us first note that due to the presence of derivative coupling, the components transforming with spin 1/2 and 3/2 may not simply be ψ_o, $\vec{\gamma}\cdot\vec{\psi}$, ψ and ϕ_k. The interaction may perhaps mix these. In order to find the correct combinations, let us write the field equations as

$$- \gamma\cdot p\psi_\mu + \frac{1}{3}(\gamma_\mu p^\nu + p_\mu \gamma^\nu)\psi_\nu - \frac{1}{3}\gamma_\mu \gamma\cdot p\gamma\cdot\psi + m(g_\mu{}^\nu - \frac{1}{3}\gamma_\mu\gamma^\nu)\psi_\nu$$

$$- (a + b\gamma_5)p_\nu\psi = 0$$

$$- (\gamma\cdot p - \lambda m)\psi - (a + b\gamma_5)p\cdot\psi = 0 \ .$$

The reason for use of this particular form will be given shortly. Here we are considering the non electromagnetic case and have set $d = -1$.

If we let underlined indices run from 0 to 4, the above equations can be written as

$$(\Lambda\cdot p - mD)_{\underline{\mu}}{}^{\underline{\nu}}\psi_{\underline{\nu}} = 0$$

where

$$(\Lambda^\sigma)_{\underline{\mu}}{}^{\underline{\nu}} = -\gamma^\sigma g_{\underline{\mu}}{}^{\underline{\nu}} + \frac{1}{3}\gamma_{\underline{\mu}} g^{\underline{\nu}\sigma} + \frac{1}{3}\gamma^{\underline{\nu}} g_{\underline{\mu}}{}^\sigma - \frac{1}{3}\gamma_{\underline{\mu}}\gamma^\sigma\gamma^{\underline{\nu}}$$

$$- (a + b\gamma_5) g_{\underline{\mu}}{}^\sigma g_4{}^{\underline{\nu}} - (a + b\gamma_5) g_{\underline{\mu}}{}^4 g^{\sigma\underline{\nu}}$$

$$(D)_{\underline{\mu}}{}^{\underline{\nu}} = - g_{\underline{\mu}}{}^{\underline{\nu}} + \frac{1}{3}\gamma_{\underline{\mu}}\gamma^{\underline{\nu}} + (1 - \lambda)\, g_{\underline{\mu}}{}^4 g_4{}^{\underline{\nu}}$$

and

$$g_{44} = g^{44} = 1 \qquad\qquad \gamma_4 = 0$$

$$g_{4\mu} = 0 \qquad\qquad \psi = \psi_4 \ .$$

The eigenvalues of the matrix $\gamma^o \Lambda^o$ solve the equation

$$x^4 (x-1)^8 [x^2 + \tfrac{5}{3}x + (\tfrac{2}{3} - g^2)]^4 = 0 \tag{5.17}$$

where

$$g^2 = a^2 + b^2 \ .$$

We may now perhaps justify the writing of the field equations in the manner used, with regard to the choice of signs. When we consider the case $g^2 = 0$, representing a non-interacting RS spin 3/2 field and a Dirac spin 1/2 field, the eigenvalues of $\gamma^o \Lambda^o$ solve, by (5.17),

$$\{x^4 (x-1)^8 (x + \tfrac{2}{3})^4\}(x+1)^4 = 0 \ .$$

The first three eigenvalues (0, 1 and -2/3) correspond to the RS field, as discussed previously. The eigenvalue -1 corresponds to the Dirac field with a negative definite charge density.

We are interested in the case $g^2 = 2/3$, when the eigenvalue equation (5.17) becomes

$$x^4 (x-1)^8 x^4 (x + \tfrac{5}{3})^4 = 0 \ .$$

We shall mention the significance of the doubling of the degeneracy of the zero eigenvalue in section D of this chapter. The projection matrices for the eigenvalues 0 and -5/3 are

$$(P_o)_{\underline{\mu}}{}^{\underline{\nu}} = \begin{pmatrix} \frac{3}{5} & 0 & 0 & 0 & -\frac{3}{5}\gamma^o(a+b\gamma_5) \\ 0 & \frac{1}{3}\gamma_1\gamma^1 & \frac{1}{3}\gamma_1\gamma^2 & \frac{1}{3}\gamma_1\gamma^3 & 0 \\ 0 & \frac{1}{3}\gamma_2\gamma^1 & \frac{1}{3}\gamma_2\gamma^2 & \frac{1}{3}\gamma_2\gamma^3 & 0 \\ 0 & \frac{1}{3}\gamma_3\gamma^1 & \frac{1}{3}\gamma_3\gamma^2 & \frac{1}{3}\gamma_3\gamma^3 & 0 \\ -\frac{3}{5}\gamma_o(a+b\gamma_5) & 0 & 0 & 0 & \frac{2}{5} \end{pmatrix}$$

and

$$(P_{-5/3})_{\underline{\mu}}{}^{\underline{\nu}} = \frac{2}{5}\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{3}{2}\gamma^o(a+b\gamma_5) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2}\gamma^o(a+b\gamma_5) & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

The spin 1/2 components are then

$$\phi \equiv \psi + (a - b\gamma_5)\gamma^o\psi_o$$

$$\tau \equiv \psi - \frac{3}{2}(a - b\gamma_5)\gamma^o\psi_o$$

and

$$\vec{\gamma}\cdot\vec{\psi}$$

while the spin 3/2 components are

$$\phi_k = [g - P_o - P_{-5/3}]_{\underline{\mu}}{}^{\underline{\nu}}\psi_{\underline{\nu}} = (g_k{}^{\ell} - \frac{1}{3}\gamma_k\gamma^{\ell})\psi_{\ell} .$$

The primary constraint (5.6) written in terms of these fields is

$$(\frac{3}{5}m - \vec{\gamma}\cdot\vec{\pi})(a + b\gamma_5)\phi + \pi^k\phi_k = \frac{3}{5}m(a + b\gamma_5)\tau . \qquad (5.18)$$

ii) Calculation of the ETACR

To determine the anticommutation relations, we start with the generator (5.11),

$$G = -i\int d^3y\{P^k\delta\phi_k - P\delta\psi\}$$

where

$$P^k = \phi^{k\dagger} - m\psi^{\dagger}(a - b\gamma_5)\bar{\Delta}\pi^k + \frac{2}{3}\phi^{\ell\dagger}\pi_{\ell}\bar{\Delta}\pi^k$$

$$P = d\psi^{\dagger} + m\phi^{k\dagger}\pi_k\bar{\Delta}(a + b\gamma_5) - \frac{2}{3}g^2\psi^{\dagger}\bar{\Delta}$$

$$\bar{\Delta} = [m^2 - \frac{4}{9}(\vec{\gamma}\cdot\vec{\pi})^2]^{-1} .$$

Since all field components in the expression for G are independent (not related by constraints), we can find the anticommutation relations directly.

We begin with

$$\delta\phi_k(x) = i[G,\phi_k(x)]$$

$$= \int d^3y[P^{\ell}\{\delta\phi_{\ell}(y),\phi_k(x)\} - \{P^{\ell}(y),\phi_k(x)\}\delta\phi_{\ell}$$

$$- P\{\delta\psi(y),\psi(x)\} + \{P(y),\phi_k(x)\}\delta\psi] .$$

As usual, we assume

$$\{\phi_\ell(x),\phi_k(y)\} = 0 = \{\psi(x),\psi(y)\} .$$

If we also assume

$$\{P(y),\phi_k(x)\} = 0 ,$$

then we obtain the relation

$$\{\psi^\dagger(y),\phi_k(x)\} = m\{\phi^{\ell\dagger}(y),\phi_k(x)\}\pi_\ell\bar{\Delta}(a+b\gamma_5)\,E^{-1} \tag{5.19}$$

where

$$E = -d + \frac{2}{3}\,g^2(\vec{\gamma}\cdot\vec{\pi})^2\,\bar{\Delta} = -d - \frac{3}{2}\,g^2 + \frac{3}{2}\,m^2g^2\bar{\Delta}$$

We are now left to solve the equation

$$\int d^3y\{\phi^{m\dagger}(y),\phi_k(x)\}A_m{}^\ell\delta\phi_\ell = -\delta\phi_k(x)$$

where

$$A_m{}^\ell = g_m{}^\ell + \frac{2}{3}\,\pi_m(d+\frac{3}{2}\,g^2)\,E^{-1}\,\bar{\Delta}\,\pi^\ell \quad .$$

The solution to this equation, which is the same problem as in equation (5.12), is

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -P_k{}^m(g_m{}^n - \frac{2}{3}\,\pi_m\Delta\pi^n)P_n{}^\ell\delta(\vec{x}-\vec{y}) \tag{5.20}$$

where

$$\Delta = [m^2/(1+\frac{3g^2}{2d}) + \frac{2e}{3}\,\vec{\sigma}\cdot\vec{B}]^{-1}$$

$$P_k{}^\ell = g_k{}^\ell - \frac{1}{3}\,\gamma_k\gamma^\ell \quad .$$

Using (5.19), we also obtain

$$\{\psi^\dagger(y),\phi_k(x)\} = \frac{m}{d}\,(g_k{}^\ell - \frac{1}{3}\,\gamma_k\gamma^\ell)\pi_\ell\Delta(1+\frac{3g^2}{2d})^{-1}\,(a+b\gamma_5)\delta(\vec{x}-\vec{y}) \quad . \tag{5.21}$$

In order to solve

$$\delta\psi(x) = i[G,\psi(x)]$$

$$= \int d^3y[P^k\{\delta\phi_k(x),\psi(x)\} - \{P^k(y),\psi(x)\}\delta\phi_k$$

$$- P\{\delta\psi(x),\psi(x)\} + \{P(x),\psi(y)\}\delta\psi] ,$$

we again assume

$$\{\phi_k(y),\psi(x)\} = 0 = \{\psi(y),\psi(x)\}$$

and

$$\{P^k(y),\psi(x)\}(g_k{}^\ell - \frac{1}{3}\gamma_k\gamma^\ell) = 0 .$$

We then obtain

$$\{\psi^\dagger(y),\psi(x)\} = \frac{1}{d}(m^2 + \frac{2e}{3}\vec{\sigma}\cdot\vec{B})\Delta(1 + \frac{3g^2}{2d})^{-1}\delta(\vec{x}-\vec{y}). \quad (5.22)$$

We can now obtain the anticommutation relations involving ψ_o. For this we use the primary constraint (5.6)

$$(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\psi_o + \pi^k\phi_k - \vec{\gamma}\cdot\vec{\pi}(a + b\gamma_5)\psi = 0 .$$

Operating on (5.20) with π^k from the left and using (5.21) gives

$$\{\phi^{\ell\dagger}(y),\psi_o(x)\} = \gamma^o[m/(1 + \frac{3g^2}{2d}) + \frac{2}{3}\vec{\gamma}\cdot\vec{\pi}]\Delta\pi^m P_m{}^\ell\delta(\vec{x}-\vec{y}). \quad (5.23)$$

Contracting (5.21) with π^k from the right and using (5.22) yields

$$\{\psi^\dagger(y),\psi_o(x)\} = -\frac{1}{d}\gamma^o[e\vec{\sigma}\cdot\vec{B} - m\vec{\gamma}\cdot\vec{\pi}]\Delta(1 + \frac{3g^2}{2d})^{-1}(a + b\gamma_5)\delta(\vec{x}-\vec{y}).$$

(5.24)

Operating on (5.23) with π_ℓ from the right and using (5.24) gives the result

$$\{\psi^{o\dagger}(y),\psi_o(x)\} = -[e\vec{\sigma}\cdot\vec{B}/(1+\frac{3g^2}{2d}) + \frac{2}{3}(\vec{\gamma}\cdot\vec{\pi})^2]\Delta\delta(\vec{x}-\vec{y}). \qquad (5.25)$$

The anticommutation relations involving the components $\vec{\gamma}\cdot\vec{\psi}$ can be obtained by use of the secondary constraint (5.7)

$$\vec{\gamma}\cdot\vec{\psi} = \gamma^o\psi_o - \frac{2ie}{3m^2}(1+\frac{3g^2}{2d})[\gamma_5\gamma\cdot\tilde{F}\cdot\psi + 2\gamma\cdot F\cdot\psi + \frac{3}{2}(a+b\gamma_5)\gamma\cdot F\cdot\gamma\psi]$$
$$+ 3\lambda(a-b\gamma_5)\psi \quad .$$

We are interested in the case when $1+\frac{3g^2}{2d}=0$, in which causal propagation was found to occur in the c-number theory. Substituting this into (5.20) through to (5.25) yields the relations

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -(g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\delta(\vec{x}-\vec{y})$$

$$\{\psi^\dagger(y),\psi_k(x)\} = -\frac{1}{m}(g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\pi_\ell(a+b\gamma_5)\delta(\vec{x}-\vec{y})$$

$$\{\psi^\dagger(y),\psi(x)\} = -(1+\frac{2e}{3m^2}\vec{\sigma}\cdot\vec{B})\delta(\vec{x}-\vec{y})$$

$$\{\phi^{\ell\dagger}(y),\psi_o(x)\} = \frac{1}{m}\gamma^o\pi^k(g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\delta(\vec{x}-\vec{y})$$

$$\{\psi^\dagger(y),\psi_o(x)\} = \frac{1}{m^2}[e\vec{\sigma}\cdot\vec{B} - m\vec{\gamma}\cdot\vec{\pi}](a+b\gamma_5)\delta(\vec{x}-\vec{y})$$

$$\{\psi^{o\dagger}(y),\psi_o(x)\} = -\frac{e}{m^2}\vec{\sigma}\cdot\vec{B}\;\delta(\vec{x}-\vec{y}) \quad . \qquad (5.26)$$

The relations in terms of the spin 1/2 fields

$$\phi = \psi + (a-b\gamma_5)\gamma^o\psi_o$$

$$\tau = \psi - \frac{3}{2}(a - b\gamma_5)\gamma^o\psi_o$$

are found to be

$$\{\phi^{\ell\dagger}(y),\phi_k(x)\} = -(g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\delta(\vec{x}-\vec{y})$$

$$\{\phi^{\ell\dagger}(y),\phi(x)\} = 0$$

$$\{\phi^{\ell\dagger}(y),\tau(x)\} = -\frac{5}{2m}(a - b\gamma_5)\pi^k(g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\delta(\vec{x}-\vec{y})$$

$$\{\phi^\dagger(y),\phi(x)\} = -\delta(\vec{x}-\vec{y})$$

$$\{\phi^\dagger(y),\tau(x)\} = -(1 - \frac{5}{3m}\vec{\gamma}\cdot\vec{\pi})\delta(\vec{x}-\vec{y})$$

$$\{\tau^\dagger(y),\tau(x)\} = -(1 + \frac{25e}{6m^2}\vec{\sigma}\cdot\vec{B})\delta(\vec{x}-\vec{y}) \quad .$$

We find these relations are consistent with the constraint (5.18)

$$(\frac{3}{5m} - \vec{\gamma}\cdot\vec{\pi})(a + b\gamma_5)\phi + \pi^k\phi_k = \frac{3}{5}m(a + b\gamma_5)\tau \quad .$$

The ETACR involving the other spin 1/2 field $\vec{\gamma}\cdot\vec{\psi}$ can be obtained using the secondary constraint (5.7) which, for $1 + \frac{3g^2}{2d} = 0$, is

$$\vec{\gamma}\cdot\vec{\psi} = \gamma^o\psi_o + 3\lambda(a - b\gamma_5)\psi \quad .$$

These anticommutators turn out to be

$$\{\phi^{\ell\dagger}(y),\vec{\gamma}\cdot\vec{\psi}(x)\} = \frac{1}{m}(1 - 2\lambda)\pi^k(g_k^{\ \ell} - \frac{1}{3}\gamma_k\gamma^\ell)\delta(\vec{x}-\vec{y})$$

$$\{\phi^\dagger(y),\vec{\gamma}\cdot\vec{\psi}(x)\} = -\frac{1}{m^2}[3m^2\lambda + (1 - 2\lambda)m\vec{\gamma}\cdot\vec{\pi}](a + b\gamma_5)\delta(\vec{x}-\vec{y})$$

$$\{\tau^{\dagger}(y),\vec{\gamma}\cdot\vec{\psi}(x)\} = \frac{1}{m^2}\left[\frac{5}{2}(1-2\lambda)\,e\vec{\sigma}\cdot\vec{B} - (1+3\lambda)m\vec{\gamma}\cdot\vec{\pi} - 3m^2\lambda\right]$$

$$\times\ (a + b\gamma_5)\,\delta(\vec{x}-\vec{y})$$

$$\{[\vec{\gamma}\cdot\vec{\psi}(y)]^{\dagger},\vec{\gamma}\cdot\vec{\psi}(x)\} = -\frac{1}{m^2}[(1-2\lambda)^2 e\vec{\sigma}\cdot\vec{B} + 6m^2\lambda^2]\,\delta(\vec{x}-\vec{y}). \quad (5.27)$$

Thus we have a fairly complex situation. The anticommutator for the "physical" spin 3/2 field ϕ_k is positive definite, while that for the independent spin 1/2 "ghost" field ϕ is negative definite. Considering only these two fields, the successful use of an indefinite metric is possible. However, the anticommutators for the two auxiliary fields τ and $\vec{\gamma}\cdot\vec{\psi}$ are each indefinite, being dependent on the external field. This will make the inclusion of an indefinite metric more difficult to handle in placing all observables in that section of the Hilbert space characterized by a positive definite metric.

Since completing this work, we have become aware of a similar set of calculations performed previously. Prabhakaran et al [1975a] found that causality was preserved in the BG system (with the coupling constant $b = 0$) if the total free charge was allowed to be indefinite. It is also claimed (see Mathews et al [1979]) in an unpublished study that the anticommutators evaluated by Schwinger's action principle are independent of the external field (J. Prabhakaran, Ph.D. thesis, Madras University 1975 [unpublished]). Our analysis shows that

only those anticommutators involving the fields ϕ_k and ϕ have this property; the dependent fields τ and $\vec{\gamma}\cdot\vec{\psi}$ still suffer from the indefiniteness problem.

C. Coupling to a Spinor and a Scalar Field

It was seen that the BG system remained causal in the presence of electromagnetic coupling if the total free charge was indefinite. As well, the indefiniteness associated with some of the anticommutation relations was found to disappear. It would be interesting to see if the same sort of effects occur for the interaction with a scalar and a spinor field. As discussed in chapter III, section B, this coupling with the RS field suffers from the same inconsistencies as found in the case of minimal coupling.

i) Investigation of Causality

We begin with the Lagrangian

$$\begin{aligned}\mathcal{L} = {} & \bar{\psi}^{\mu}(\Gamma\cdot p - B)_{\mu}{}^{\nu}\psi_{\nu} + d\bar{\psi}(\gamma\cdot p - \lambda m)\psi + \bar{\chi}(\gamma\cdot p - M)\chi \\ & - \frac{1}{2}(p^{\nu}\rho p_{\nu}\rho + \mu^2\rho^2) - \bar{\psi}^{\mu}(a + b\gamma_5)p_{\mu}\psi - \bar{\psi}(a + b\gamma_5)p^{\mu}\psi_{\mu} \\ & - f\bar{\psi}^{\mu}\theta_{\mu\nu}\chi p^{\nu}\rho - f\bar{\psi}\gamma_{\nu}\chi p^{\nu}\rho - f(p^{\nu}\rho)\bar{\chi}\theta_{\nu\mu}\psi^{\mu} - f(p^{\nu}\rho)\bar{\chi}\gamma_{\nu}\psi. \qquad (5.28)\end{aligned}$$

Here, the scalar and spinor fields are ρ and χ, respectively, with $\theta_{\mu\nu}$ given by

$$\theta_{\mu\nu} = g_{\mu\nu} - h\gamma_{\mu}\gamma_{\nu} \quad .$$

Prabhakaran et al [1975b] found that causality is preserved, for $b=0$, if the choice $d=-\frac{3}{2}g^2$ is made. This is the same condition as was found in the case of minimal coupling, and necessarily makes the total free charge density of the BG system indefinite. We will perform the same calculation to see if inclusion of the γ_5 term has any effect.

The equations of motion following from the Lagrangian (5.28) are

$$(\gamma\cdot p - m)\psi_\mu - \frac{1}{3}(\gamma_\mu p^\nu + p_\mu\gamma^\nu)\psi_\nu + \frac{1}{3}\gamma_\mu(\gamma\cdot p + m)\gamma\cdot\psi$$
$$+ (a + b\gamma_5)p_\mu\psi + f\theta_{\mu\nu}\chi p^\nu\rho = 0 \tag{5.29}$$

$$d(\gamma\cdot p - \lambda m)\psi - (a + b\gamma_5)p\cdot\psi - f\gamma_\nu\chi p^\nu\rho = 0 \tag{5.30}$$

$$(\gamma\cdot p - M)\chi - f(p^\nu\rho)\theta_{\nu\mu}\psi^\mu = 0 \tag{5.31}$$

$$(p^2 - \mu^2)\rho + fp^\nu[\bar{\psi}^\mu\theta_{\mu\nu}\chi + \bar{\psi}\gamma_\nu\chi + \bar{\chi}\theta_{\nu\mu}\psi^\mu + \bar{\chi}\gamma_\nu\psi] = 0 . \tag{5.32}$$

Constraints are present in (5.29). Contracting this equation with γ^μ and p^μ yields, respectively,

$$p\cdot\psi + \frac{m}{2}\gamma\cdot\psi + \frac{3}{2}(a - b\gamma_5)\gamma\cdot p\psi + \frac{3}{2}f\gamma^\mu\theta_{\mu\nu}\chi p^\nu\rho = 0 \tag{5.33}$$

$$\frac{2}{3}\gamma\cdot pp\cdot\psi - mp\cdot\psi + \frac{1}{3}m\gamma\cdot p\gamma\cdot\psi + (a + b\gamma_5)p^\nu\psi$$
$$+ fp^\mu(\theta_{\mu\nu}\chi p^\nu\rho) = 0 . \tag{5.34}$$

The primary constraint results by taking the zeroth component of (5.29) and using (5.33). The result is

$$[\vec{p} + \frac{1}{3}\vec{\gamma}\cdot\vec{p}\vec{\gamma}]\cdot\vec{\psi} + \vec{\gamma}\cdot\vec{p}(a + b\gamma_5)\psi = f\gamma^k\theta_{k\nu}\chi p^\nu\rho + (m - \frac{2}{3}\vec{\gamma}\cdot\vec{p})\gamma^o\psi_o \ .$$

In order for constraints to emerge from this, we must have $\theta_{ko} = 0$. We then conclude $h = 0$, and the primary constraint becomes

$$(m - \frac{2}{3}\vec{\gamma}\cdot\vec{p})\gamma^o\psi_o + p^k\phi_k = \vec{\gamma}\cdot\vec{p}(a + b\gamma_5)\psi + f\vec{\gamma}\cdot\chi\vec{p}\rho \quad . \qquad (5.35)$$

Multiplying (5.33) by $\frac{2}{3}\gamma\cdot p$ and subtracting this from (5.34) yields the relation

$$p\cdot\psi = \frac{f}{m}[\gamma_\mu\gamma\cdot p - p_\mu]\chi p^\mu\rho \quad . \qquad (5.36)$$

To find the secondary constraint, solve for $\gamma\cdot p\psi$ in (5.30) and substitute in (5.33). We obtain

$$\gamma\cdot\psi = -\frac{2}{m}\{\frac{f}{m}(1 + \frac{3g^2}{2d})p\cdot\psi + \frac{3f}{2d}(a - b\gamma_5)\gamma_\nu\chi p^\nu\rho + \frac{3}{2}\lambda m(a - b\gamma_5)\psi$$
$$+ \frac{3}{2}f\gamma_\mu\chi p^\mu\rho\}$$

where $g^2 = a^2 + b^2$ and $p\cdot\psi$ is given by (5.36).

We wish to see if there is causal propagation for $d = -\frac{3}{2}g^2$. Assuming this relation, we have

$$\gamma\cdot\psi = -3\lambda(a - b\gamma_5)\psi - \frac{3f}{m}\gamma^\mu\chi p_\mu\rho - \frac{3f}{md}(a - b\gamma_5)\gamma^\nu\chi p_\nu\rho \ . \qquad (5.37)$$

We again employ the shock wave formalism of Madore and Tait [1973] (see Appendix II). Denoting the magnitude of

the discontinuity across a characteristic surface by a square bracket, we have

$$[p_\mu \psi_\nu] = n_\mu K_\nu$$

$$[p_\mu \psi] = n_\mu R$$

$$[p_\mu \chi] = n_\mu S$$

$$[p_\mu p_\nu \rho] = n_\mu n_\nu T$$

and $\quad [\psi_\mu] = [\psi] = [\chi] = [p_\mu \rho] = 0 \; .$

In order to see if there is acausal propagation, assume in the following that $n^2 \neq 0$.

Taking the discontinuity of (5.31) and (5.30), we find

$$(\gamma \cdot n) S = 0 \implies S = 0$$

and $\quad d\gamma \cdot nR = (a + b\gamma_5) n \cdot K \; .$

By considering (5.36), we have

$$n \cdot K = \frac{f}{m} [\gamma_\mu \gamma_\nu \chi (n^\nu n^\mu T) - \chi n^2 T] = 0$$

from which we conclude $R = 0$.

Taking the discontinuity of (5.32), we obtain

$$n^2 T = 0 \implies T = 0 \; .$$

Differentiating (5.37) and then taking the discontinuity leads to

$$\gamma \cdot K = 0 \; .$$

Finally, taking the discontinuity of (5.29) gives

$$\gamma \cdot n K_\mu = 0$$

from which we conclude $K_\mu = 0$. Thus, all discontinuities vanish for $n^2 \neq 0$. The characteristic surfaces are therefore those of the lightcone, and propagation is causal.

ii) Calculation of the ETACR

It now remains to be seen how the commutation relations are affected in the BG theory with a scalar-spinor coupling subject to the condition $d = -\frac{3}{2} g^2$. However, by considering the Lagrangian (5.28) and the complicated nature of the primary constraint (5.35), the task will be quite involved.

We can simplify matters extensively by using a method, for example, employed by Hagen [1971]. Consider the free spin zero Lagrangian

$$\mathcal{L}_o = -\frac{1}{2} [p^\mu \rho p_\mu \rho + \mu^2 \rho^2] \quad .$$

This Lagrangian is equivalent to

$$\mathcal{L}_o = [\rho p^\mu \rho_\mu - \rho^\mu p_\mu \rho] + \frac{1}{2} \rho^\mu \rho_\mu - \frac{1}{2} \mu^2 \rho^2$$

where ρ and ρ^μ are considered to be independent fields. This can be seen by finding the field equations implied by this Lagrangian. Varying with respect to ρ^μ gives

$$\rho_\mu = p_\mu \rho$$

while varying with respect to ρ gives

$$p^{\mu}\rho_{\mu} = \mu^2 \rho \quad .$$

Substituting in for ρ_{μ} in this last equation gives

$$(p^2 - \mu^2)\,\rho = 0$$

which is the same field equation as implied by the original Lagrangian. Hence, both are equivalent.

The coupling of the BG field to the spinor and scalar field can be effected by addition of the interaction term

$$\mathcal{L}_{int} = - f\, j^{\mu} \rho_{\mu} - f\rho^{\mu} (j_{\mu})^{\dagger}$$

where

$$j^{\mu} = \bar{\psi}^{\mu}\chi + \bar{\psi}\gamma^{\mu}\chi \quad .$$

Considering ρ and ρ^{μ} as independent fields, no derivative terms appear explicitly in $\mathcal{L}_{int}$. The total Lagrangian is then

$$\mathcal{L} = \bar{\psi}^{\mu}(\Gamma\cdot p - B)_{\mu}{}^{\nu}\psi_{\nu} + d\bar{\psi}(\gamma\cdot p - \lambda m)\psi + \bar{\chi}(\gamma\cdot p - M)\chi$$
$$- (\rho^{\mu}p_{\mu}\rho - \rho p^{\mu}\rho_{\mu}) + \frac{1}{2}\rho^{\mu}\rho_{\mu} - \frac{1}{2}\mu^2\rho^2$$
$$- \bar{\psi}^{\mu}(a + b\gamma_5)p_{\mu}\chi - \bar{\psi}(a + b\gamma_5)p^{\mu}\psi_{\mu} - fj^{\mu}\rho_{\mu} - f\rho^{\mu}(j_{\mu})^{\dagger} \quad .$$

The kinetic energy terms (those containing time derivatives) are

$$\mathcal{L}_k = -\phi^{k\dagger}p_o\phi_k - \frac{2}{3}\psi^{o\dagger}p_o\psi_o + d\psi^{\dagger}p_o\psi + \chi^{\dagger}p_o\chi - \rho^{o}p_o\rho + \rho p_o\rho_o$$
$$-\bar{\psi}^{o}(a + b\gamma_5)p_o\psi - \bar{\psi}(a + b\gamma_5)p_o\psi_o \quad .$$

Since we wish to examine the anticommutation relations for the case $d=-\frac{3}{2}g^2$, we assume this condition from the outset. We can then write $\mathcal{L}_k$ as, with $d=-1$,

$$\mathcal{L}_k = -\phi^{k\dagger} p_o \phi_k - \phi^\dagger p_o \phi + \chi^\dagger p_o \chi - \rho^o p_o \rho + \rho p_o \rho_o$$

where

$$\phi = \psi + (a - b\gamma_5)\gamma^o \psi_o \quad .$$

The generator can now be written down directly:

$$G = -i\int d^3y \{\phi^{k\dagger}\delta\theta_k + \phi^\dagger\delta\phi - \chi^\dagger\delta\chi + \rho^o\delta\rho - \rho\delta\rho_o\} \; .$$

Recalling that $\phi(x)$ is an independent field, the anticommutation relations can now be derived with relative ease. The variation of a field component $W(x)$ is given by

$$\delta W(x) = i[G,W(x)]$$
$$= \int d^3y[\phi^{k\dagger}\delta\phi_k + \phi^\dagger\delta\phi - \chi^\dagger\delta\chi + \rho^o\delta\rho - \rho\delta\rho_o, W(x)] . \quad (5.38)$$

In the following, we shall assume that the equal-time anticommutators of any of the fields $\phi_k(y)$, $\phi(y)$, or $\chi(y)$ with $\phi_\ell(x)$, $\phi(x)$ or $\chi(x)$ vanish.

With $W(x) = \rho(x)$ in (5.38), we find

$$[\phi^{k\dagger}(y),\rho(x)] = 0$$

$$[\phi^\dagger(y),\rho(x)] = 0$$

$$[\chi^\dagger(y),\rho(x)] = 0$$

$$[\rho(y),\rho(x)] = 0$$

$$[\rho^o(y),\rho(x)] = \delta(\vec{x}-\vec{y}) \implies [\dot{\rho}(y),\rho(x)] = -i\delta(\vec{x}-\vec{y}) \; .$$

For $W(x) = \rho^{o}(x)$, we obtain

$$[\phi^{k\dagger}(y), \rho^{o}(x)] = 0$$

$$[\phi^{\dagger}(y), \rho^{o}(x)] = 0$$

$$[\chi^{\dagger}(y), \rho^{o}(x)] = 0$$

$$[\rho^{o}(y), \rho^{o}(x)] = 0$$

$$-[\rho(y), \rho^{o}(x)] = \delta(\vec{x}-\vec{y}) \implies [\dot{\rho}(x), \rho(y)] = -i\delta(\vec{x}-\vec{y}).$$

With $W(x) = \phi(x)$, we get

$$\{\phi^{k\dagger}(y), \phi(x)\} = 0$$

$$\{\phi^{\dagger}(y), \phi(x)\} = -\delta(\vec{x}-\vec{y})$$

$$\{\chi^{\dagger}(y), \phi(x)\} = 0$$

$$[\rho^{o}(y), \phi(x)] = 0$$

$$[\rho(y), \phi(x)] = 0 \; .$$

For $W(x) = \chi(x)$, we find

$$\{\phi^{k\dagger}(y), \chi(x)\} = 0$$

$$\{\phi^{\dagger}(y), \chi(x)\} = 0$$

$$\{\chi^{\dagger}(y), \chi(x)\} = \delta(\vec{x}-\vec{y})$$

$$[\rho^{o}(y), \chi(x)] = 0$$

$$[\rho(y), \chi(x)] = 0 \; .$$

Finally, for $W(x) = \phi_\ell(x)$, we obtain

$$
\begin{aligned}
\{\phi^{k\dagger}(y), \phi_\ell(x)\} &= -(g_\ell{}^k - \tfrac{1}{3}\gamma_\ell\gamma^k)\delta(\vec{x}-\vec{y}) \\
\{\phi^\dagger(y), \phi_\ell(x)\} &= 0 \\
\{\chi^\dagger(y), \phi_\ell(x)\} &= 0 \\
[\rho^o(y), \phi_\ell(x)] &= 0 \\
[\rho(y), \phi_\ell(x)] &= 0
\end{aligned}
\tag{5.39}
$$

[The particular form of the anticommutator between $\phi^{k\dagger}(y)$ and $\phi_\ell(x)$ was chosen so as to satisfy $\gamma^k\phi_k = 0$. Notice that $(g_k{}^m - \frac{1}{3}\gamma_k\gamma^m)(g_m{}^n - \frac{1}{3}\gamma_m\gamma^n) = g_k{}^n - \frac{1}{3}\gamma_k\gamma^n$].

These are certainly a more desirable set of relations than the corresponding ones found by Hagen [1971] for the RS field. The fields $\chi(x)$ and $\rho(x)$ satisfy their free field commutation relations. The anticommutator for the spin 3/2 field $\phi_k(x)$ is positive definite, while the "ghost" field $\phi(x)$ has a negative definite anticommutator.

However, let us see what the anticommutator becomes for the spin 1/2 field

$$\tau = \psi - \frac{3}{2}(a - b\gamma_5)\gamma^o\psi_o \quad .$$

By use of the primary constraint (5.35),

$$\frac{3}{5}m(a + b\gamma_5)\tau = p^k\phi_k + (\frac{3}{5}m - \vec{\gamma}\cdot\vec{p})(a + b\gamma_5)\phi - f\vec{\gamma}\cdot\chi\vec{p}\rho$$

we can derive the relation

$$\{\tau^{\dagger}(y), \tau(x)\} = -\ [1 - \frac{25f^2}{6m^2} (\vec{p}\rho)^2] \delta(\vec{x}-\vec{y}) \ .$$

Thus, we find this anticommutator is indefinite, just as was found in the case of electromagnetic coupling, where

$$\{\tau^{\dagger}(y), \tau(x)\} = -\ [1 + \frac{25e}{6m^2} \vec{\sigma}\cdot\vec{B}]\ \delta(\vec{x}-\vec{y}) \ .$$

To find the complete set of commutation relations involving $\tau(x)$ and $\vec{\gamma}\cdot\vec{\psi}(x)$ would require the action principle to be applied for the independent fields ρ, ρ^o, ϕ_k, χ and ψ, with subsequent use of the primary constraint (5.35) and secondary constraint (5.37). However, from noting the above relation obtained for $\tau(x)$, we would expect a similar set to that obtained in the case of minimal coupling (see 5.27). It seems safe to conclude that for the scalar-spinor interaction, just as was found for minimal coupling, the indefiniteness of the anticommutation relations for the independent fields $\phi_k(x)$ and $\phi(x)$ disappears but remains for the two spin 1/2 fields $\tau(x)$ and $\vec{\gamma}\cdot\vec{\psi}(x)$.

D. Significance of the Choice $d = -\frac{3}{2}\, g^2$

In light of the preceding work in this chapter, it would be interesting to know the reason why the causality problem and indefiniteness of the anticommutator of the independent field ϕ_k both disappear with the choice $d = -\frac{3}{2}\, g^2$. In this section we shall attempt to answer this

question for the case of minimal coupling.

We can gain an insight into the significance of the particular choice of the coupling constant if we examine the constraints present in the theory. Here we are considering equations of the form

$$(\Lambda\cdot\pi - mD)\psi = 0 \ . \qquad (5.40)$$

Recall from the discussion of constraints in chapter II (see pages 4-9) that primary constraints emerge from the singular nature of the matrix $\gamma^o\Lambda_o$. As well, there must exist secondary constraints on the fields for spin $s \geq 3/2$ if the quantization is to be done consistently on a positive definite metric. Now, when the matrix $\gamma^o\Lambda_o$ is found for the BG system (see pages 52-55), it was discovered that the degeneracy of the zero eigenvalue doubled for $d = -\frac{3}{2}g^2$. From the form of the projection matrix P_o in this case, it is seen that not one but two independent constraint equations will emerge when P_o is operated on (5.40). One such constraint is the primary constraint given by (5.6),

$$(m - \frac{2}{3}\vec{\gamma}\cdot\vec{\pi})\gamma^o\psi_o + \pi^k\phi_k - \vec{\gamma}\cdot\vec{\pi}(a + b\gamma_5)\psi = 0 \ .$$

An additional primary constraint is actually the secondary constraint given by (5.7),

$$\gamma\cdot\psi = -3\lambda(a - b\gamma_5)\psi \quad .$$

That is, the secondary constraint becomes a primary

constraint for the choice $d = -\frac{3}{2} g^2$. To see this, note that we can derive this latter constraint without any differentiation of the equations of motion simply by substituting (5.4) into (5.3) and assuming $d = -\frac{3}{2} g^2$. The absence of secondary constraints thus makes a consistent quantization on a positive definite metric impossible, as was found.

The reason that causality is preserved in this situation can be understood in view of the work of Amar and Dozzio [1975]. In this paper, it was found that causal propagation for equations of the form

$$(\alpha \cdot \pi - m)\psi = 0 \tag{5.41}$$

exists if α^o satisfies the minimal equation

$$\alpha^o \{ \prod_{i=1}^{m} [(\alpha^o)^2 - (x_i)^2]^{r_i} \} = 0 \quad . \tag{5.42}$$

Here, x_i are the non-zero eigenvalues of the matrix α^o and r_i is some positive integer. Following Amar and Dozzio [1975], we can classify the three types of constraints associated with α^o as

(a) the nondiagonalizability of the block corresponding to non-zero eigenvalues;

(b) the singularity of α^o, if α^o is diagonalizable;

(c) the nondiagonalizability of the block corresponding to the zero eigenvalues.

The condition for causal propagation (5.42) means that, in the normal Jordan form of α^o, the block of the null eigen-

values is required to be diagonalized. Thus, constraints (a) and (b) do not prevent causal propagation, but constraints (c) can lead to difficulties. This, however, does not mean that constraints of type (c) always lead to acausal propagation. In the case of the RS field, where α^o satisfies $(\alpha^o)^2[(\alpha^o)^2 - 1] = 0$, they do. However, Capri and Shamaly [1972] gave an example of a causal spin 1 theory whose minimal equation is $(\alpha^o)^3[(\alpha^o)^2 - 1] = 0$, and thus contains type (c) constraints. Thus, equations with a nondiagonalizable null eigenvalue block cannot be ruled out on the grounds of causality.

In the case of the BG theory, the 20×20 matrix α^o has the form

$$(\alpha^o)_\mu{}^\nu = \gamma^o g_\mu{}^\nu - \gamma_\mu g^{o\nu} - \frac{1}{2}\gamma^\nu g_\mu{}^o + \frac{1}{3}\gamma_\mu\gamma^o\gamma^\nu$$

$$(\alpha^o)_\mu{}^4 = (a + b\gamma_5)(g_\mu{}^o - \gamma_\mu\gamma^o)$$

$$(\alpha^o)_4{}^\nu = \frac{1}{d\lambda}(a + b\gamma_5)\, g^{\nu o}$$

$$(\alpha^o)_4{}^4 = \frac{1}{\lambda}\gamma^o \quad .$$

The eigenvalues of α^o are 0, ±1, and $\pm 1/\lambda$, irrespective of the constants a, b or d. The BG theory with $d > 0$ allows for acausal propagation, as was observed (see equation 5.9). In fact, the block of the null eigenvalues of α^o for $d > 0$ is nondiagonalizable (see Gupta [1954]) and so type (c) constraints are present.

On the other hand, for $d = -\frac{3}{2} g^2$, it is found that α^o satisfies the minimal equation

$$\alpha^o[(\alpha^o)^2 - 1][(\alpha^o)^2 - (\tfrac{1}{\lambda})^2] = 0 .$$

Thus, the block of null eigenvalues becomes diagonalizable when $d = -\frac{3}{2} g^2$, and type (c) constraints disappear. The condition for causal propagation (5.42) is then satisfied, and the propagation is causal.

We can now also see why the anticommutator of the field ϕ_k is independent of the external field. Cox [1976] considered theories with field equations of type (5.41). He showed that for cases in which α^o is diagonalizable, the anticommutator of the independent field components does not involve the external field. This is again not to suggest that type (c) constraints always lead to indefinite anticommutation relations, but that those theories containing only type (a) and (b) constraints lead to no such immediate problems. For the BG field with $d = -\frac{3}{2} g^2$, we indeed did find the ETACR of the independent fields $\phi_k(x)$ and $\phi(x)$ to be independent of the external field (see page 59). However, the dependent field components $\tau(x)$ and $\vec{\gamma}\cdot\vec{\psi}(x)$, not discussed by Cox, did have ETACR which depended on the external field and were indefinite.

Hence, theories in which α^o is diagonalizable have causal propagation and the anticommutation relations for the independent field components are not dependent on the

external field. However, the anticommutation relations for the dependent field components may still suffer from the indefiniteness problem. The BG field with indefinite total free charge density is an example where the latter effect occurs. It would be difficult to construct a consistent indefinite metric theory for the BG field due to the indefinite anticommutators still present.

CHAPTER V

CONCLUSIONS

Let us summarize the results obtained. We began with the free RS spin 3/2 Lagrangian. This is a Lorentz covariant theory, meaning that expressions derived from this Lagrangian assume the same form in all observer frames. However, when minimal coupling is introduced, two inconsistencies immediately arise. Interpreted as a c-number theory, propagation outside the lightcone is found to exist for a finite electromagnetic field. In the quantized version, certain anticommutators, which should be positive definite by definition, could in fact become negative for a sufficiently strong electromagnetic field. Thus, both postulates of special relativity - no propagation of signals can occur outside the lightcone and results obtained are independent of the particular Lorentz frame used - are seen to be violated. Minimal coupling is not unique in this sense, as the same effects were noted for the scalar-spinor interaction. The requirement that a theory be Lorentz covariant is therefore not sufficient to guarantee that the postulates of special relativity are satisfied.

The close connection between the problems of acausality and the indefiniteness of the anticommutators was brought out again in the two interactions considered in

chapter IV. This connection is further seen in chapter V, where it was found that causal propagation could occur if an indefinite total free charge is assumed for the BG field. This requires the use of an indefinite metric upon quantization. Although the anticommutators of the independent fields are not indefinite, those of the other two spin 1/2 fields are. This makes the successful use of an indefinite metric in quantizing the BG field difficult to formulate.

These effects have also been observed in at least two other formulations of spin 3/2 theories. In the Fisk-Tait theory [1973], an antisymmetric tensor spinor $\psi_{\mu\nu}$ is the basic field operator. Causality is preserved when electromagnetic and scalar-spinor interactions are included, but the total free charge is again indefinite (see Mathews et al [1979] and the references listed there). A system of equations that describe particles of arbitrary spin without subsidiary conditions has been given by Bhabha ([1945], [1949]). (The field referred to as the BG field in this work is not among this class of equations). This system remains causal with minimal coupling, but an indefinite metric is needed for quantization (Krajcik and Nieto [1976]).

The inconsistencies present in field theories of spin 3/2 also occur in other higher spin theories. As mentioned in the last paragraph, an indefinite metric is needed for quantization of the Bhabha system of equations.

At the c-number level, Velo and Zwanziger [1969b] showed that a spin 1 field with an electric quadrupole interaction violates causality. As well, a minimally coupled spin 2 field also has acausal modes of propagation. The whole area of field theories with spin $s \geq 3/2$ is beset with problems, the spin 1 case being marginal.

Given that we violate the postulates of special relativity when certain physically reasonable interactions are included in higher spin theories, it seems reasonable to attempt to give a physical explanation for the inconsistencies present in these theories.

Although we can describe a free particle of arbitrary spin with no difficulty, we must ultimately be able to derive quantities from the theory that can be verified experimentally. This being the main purpose of theoretical physics, we then have to include interaction terms, which are always present in the real world. Including such terms in the higher spin theories would seem to be a straightforward extension to the method used in the cases of lower spin, where results thus obtained are fairly accurate. The problems of acausality, indefinite anticommutators, and the possible need of an indefinite metric encountered in higher spin interacting theories all draw a clear distinction between particles of spin greater than and lower than 3/2. This distinction may indeed imply that there are no elementary particles of spin $s \geq 3/2$.

This view is supported by a work of Capri and Shamaly [1976]. Here, it is shown that if we write the equation of motion for a field ψ in a pure spin state,

$$p_o\psi = H\psi \quad ,$$

then H will be local for the minimally coupled spin 0, 1/2, and 1 cases. However, H is non-local even in the free spin 3/2 case. This could be interpreted to mean that the spin 3/2 particle has structure and is therefore not elementary. The equations considered so far have assumed that higher spin particles were elementary, and the problems associated with the interacting field cases could be explained by denial of this initial assumption. In view of the fact that all leptons discovered so far are of spin 1/2 and all hadrons seem to be composed of spin 1/2 quarks, this seems to be a physically reasonable explanation for the inconsistencies.

Taking the point of view that we can extract more information from a theory than was originally put in, the problems associated with interacting higher spin theories suggest that no such particle exists in nature, where interactions are always present. However, a lingering doubt may still persist in some minds. If a higher spin lepton were to be discovered in the future, our whole understanding of the mechanisms behind including interactions for a particle of arbitrary spin would be placed in serious question.

APPENDIX I

In this appendix we shall give some basic properties of spinors. For a more thorough treatment of spinors and relativistic wave equations, see Corson [1955] and Umezawa [1956].

A spinor is defined on a two dimensional spin space. Rotations in this space are described by the transformation laws

$$\psi^A \longrightarrow S^A{}_B \psi^B$$

$$\psi_{\dot{A}} \longrightarrow \psi_{\dot{B}} \bar{S}^B{}_A$$

where the bar over $S^B{}_A$ denotes complex conjugation. The indices A, B run from 1 to 2. These can be raised and lowered by means of the metric ϵ_{AB}.

$$\psi_A = \epsilon_{AB}\psi^B \qquad\qquad \psi_{\dot{A}} = \epsilon_{\dot{A}\dot{B}}\psi^{\dot{B}}$$

$$\psi^B = \psi_A \epsilon^{AB} \qquad\qquad \psi^{\dot{B}} = \psi_{\dot{A}}\epsilon^{\dot{A}\dot{B}}$$

where

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\dot{A}\dot{B}} \quad .$$

A four-vector x^μ can be associated with a spinor $\chi_{\dot{A}B}$ by

$$x^\mu = \frac{1}{2}\sigma^{\mu\dot{A}B}\chi_{\dot{A}B}$$

$$\chi_{\dot{A}B} = \sigma_{\mu\dot{A}B}x^\mu$$

where $\sigma^{\mu\dot{A}B}$ are the usual Pauli spin matrices

$$\sigma_{0\dot{A}B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_{1\dot{A}B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_{2\dot{A}B} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_{3\dot{A}B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In particular, for $p_\mu = i\partial_\mu$ we have

$$p_{\dot{A}B} = p_\mu \sigma^\mu{}_{\dot{A}B}$$

$$p_\mu = \frac{1}{2}\, \sigma_{\mu\dot{A}B}\, p^{\dot{A}B}$$

and

$$p^\mu p_\mu = \frac{1}{2}\, p_{\dot{A}B} p^{\dot{A}B} \quad .$$

If we define

$$S_{\mu\nu}{}^A{}_C = \frac{1}{2}\, [\sigma_\mu{}^{\dot{B}A} \sigma_{\nu\dot{B}C} - \sigma_\nu{}^{\dot{B}A} \sigma_{\mu\dot{B}C}] \ ,$$

then a symmetric spinor f_{AB} can be associated with an antisymmetric tensor $F_{\mu\nu}$ by

$$f_{AB} = \frac{1}{8}\, S_{\mu\nu AB}\, F^{\mu\nu}$$

$$F_{\mu\nu} = S_{\mu\nu}{}^{AB} f_{AB} + \bar{S}_{\mu\nu}{}^{AB} \bar{f}_{AB} \quad .$$

APPENDIX II

This is a short outline of the use of characteristic surfaces in determining if acausal propagation is present in a system or not. For a rigorous treatment of the method, see Courant and Hilbert [1962].

Consider a hyperbolic system of differential equations. The system is supplemented by certain initial conditions on the solution. A characteristic surface may be defined as those initial surfaces for which the initial value problem has no unique solution. In Courant and Hilbert [1962], it is shown that the maximum speed of propagation of signals described by the hyperbolic equations is given by the slope of the characteristic surfaces.

We illustrate the method by a hyperbolic system of first order,

$$(\Gamma^{\sigma} p_{\sigma} - B)_{\alpha}^{\ \beta} \psi_{\beta} = 0 \ .$$

Here, $\sigma = 0,1,2,3$ while α and β assume values $1,2,\ldots N$, where N is the number of equations present. The term B contains no derivatives. The normals to the characteristic surfaces, denoted by a four-vector n_{μ}, can be shown to satisfy the equation

$$D(n) \equiv |(\Gamma^{\mu} n_{\mu})_{\alpha\beta}| = 0 \ .$$

$D(n)$ is called the characteristic determinant, while $D(n) = 0$ is the characteristic equation. For causal propagation, no timelike normal must satisfy this equation,

as the characteristic surfaces would then have spacelike tangents and propagation would occur outside the lightcone.

An equivalent way for determining if acausal propagation is present has been given by Madore and Tait [1973]. This method uses the fact that, across a characteristic surface, there exist discontinuities in the highest order derivative of the solution of the wave equation (as well as all those of higher order). Thus, for our example of a first order system,

$$(\Gamma^\sigma p_\sigma - B)_\alpha^{\ \beta} \psi_\beta = 0 \ ,$$

we have

$$[\psi_\beta] = 0$$

$$[p_\mu \psi_\beta] = n_\mu K_\beta \ .$$

The square bracket denotes the magnitude of the discontinuity across the characteristic surface. The normals to the surface are again represented by n_μ, while K_β is a differentiable function of the coordinates. The discontinuity of the second and higher order derivatives of ψ_β across a characteristic surface becomes an increasingly lengthy expression (see Madore and Tait [1973]). The discontinuities of the derivatives of the solution will propagate along the characteristic surface.

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